Advanced Concepts In Fluid Mechanics
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Lecture - 37
Lubrication Theory

In the previous chapters we have discussed about the steady and unsteady flows typically in the low Reynolds number regime. It is also important to recognize that the kinds of flows that we have considered so far have a fixed confinement height or a fixed confinement diameter. But in reality there can be situations when the confinement height itself can be a function of position and time. It is an interesting situation which is schematically represented in figure 1(i).

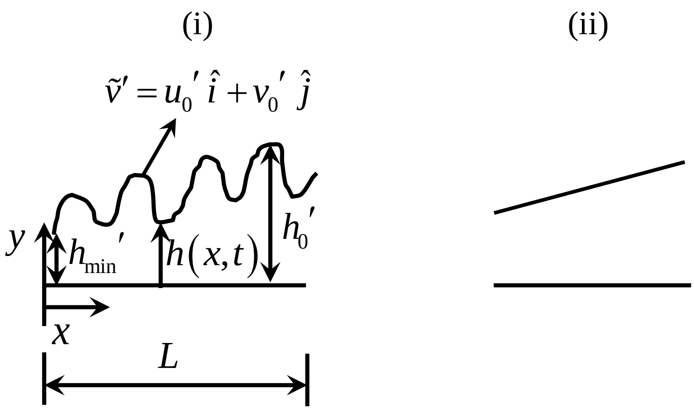

Figure 1. (i) The confinement height h as a function of position and time $h(x,t)$. (ii) Slider bearing, the configuration of which is like a tapered geometry.

Let us assume that we have a bottom boundary and there is the confinement height h which is a function of x and time. h as a function of x is quite clear but h is a function of time in a sense that the gap can be lifted or can be put down so that the gap can be driven as a function of time. One can consider this configuration as a channel. This is a special type of channel flow but the channel is not the usual channel that we are talking about. So question arises about what can be the possible applications of this kind of a scenario. We have already talked about bearings. In engineering there is a type of bearing called as slider bearing where the configuration is something like that drawn in figure 1(ii). In this case it is a smooth linear function instead of the complicated function as observed in figure 1(i). In this case (i.e. in figure 1(ii)), it is like a tapered geometry and not a regular parallel geometry. Let us take another example where we have one plate and we want to deposit a coating on this plate. We want to put another plate or a paintbrush and deposit a coating of paint on top of this. The paintbrush is coming close to the plate and trying to deposit the pattern. In a similar way one can think as if the top boundary of the channel (as depicted in figure 1(i)) is coming down. Interestingly, it is quite intuitive that when the top boundary comes down there will be a resistance to it against coming further down. When it goes up there is a resistance against it of going further up. At the end of this chapter we will learn the reason about why this is happening. All these physical problems fall under the paradigm of one mathematical theory which is called as lubrication theory. This has evolved from a pure mathematical theory based on asymptotic analysis. But it has been given subsequently an engineering flavor subsequently because the applications of this theory have been found to be very relevant in lubrication and bearing which is one of the very important engineering aspects.

For simplification, we will first consider that it is a two-dimensional problem. The *x*-axis and the *y*-axis are shown in figure 1(i). Here the height *h* is a continuous function of *x* and time but it could be a discrete function of *x* and time. Here we have the extreme heights h_0 and $h_{\min}$ where h_0 is the maximum height and $h_{\min}$ is the minimum height. The length of the channel is *L* where we assume for the time being that there is no difference in wettability across the surface. So the physics remain invariant along the length *L*. Otherwise *L* no more remains the characteristic length scale of the system along *x*. Now for this theory to work, first we have to understand the assumptions that we are going to make to get into this theory. For the theory to work, the gap height (*h*) must be much less than the length *L*. This is very similar to the low Reynolds number hydrodynamics that we have studied earlier. Here the additional complexity is the variable nature of the gap. Here we assume that the gap height is much less than the length and therefore, one question definitely comes that whether h_0 or $h_{\min}$ to be considered as the characteristic height. The answer is we should consider h_0 or the maximum value of *h* to be the characteristic height because of h_0 is much less than the length *L* then $h_{\min}$ will surely be much less than *L*. So, the conservative parameter is $\frac{h_0}{L}$ which is much less than 1. We define $\frac{h_0}{L}$ as a small parameter ε, i.e. $\varepsilon = \frac{h_0}{L} \ll 1$.

Now we will make certain assumptions. First of all we will assume that it is an incompressible flow. It is an incompressible flow and it is a two-dimensional flow also. So it is a two-dimensional incompressible flow with the continuity equation given by $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Like the previous problems or the previous cases here also we will make an order of magnitude analysis. The term $\frac{\partial u}{\partial x}$ will be of the order of $\sim\frac{u_c}{L}$ where u_c is the characteristic velocity scale and L is the characteristic length scale along x. Now question arises about the characteristic velocity scale and how it is generated. Now we will make a change in the notation of the variables. Since we will be mainly using the non-dimensional equations, we denote all the variables of the dimensional equation by the $'$ symbol; the continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ is being replaced by $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$. Since majority of the equations are in the dimensionless form, it will be easier to write. Let us assume the boundary has an arbitrary velocity $\tilde{v}'=u_0{}'\,\hat{i}+v_0{}'\,\hat{j}$ (as shown in figure 1(i)). If we give the boundary an arbitrary velocity we can generate all types of very complex flows. Not only complex flows but very complex body forces can also be present. So this problem is a very general class of problem and far more general than the flow through a channel or a pipe. So, $\frac{\partial u'}{\partial x'}$ is of the order of $\sim\frac{u_c}{L}$ and $\frac{\partial v'}{\partial y'}$ is of the order of $\sim\frac{v_c}{h_0}$. Now, here the problem is that we do not know about the velocity scales u_c and v_c because this is absolutely an arbitrary general situation which we have introduced. So u_c and v_c will depend on the actual physics which is governing the problem. We cannot really comment anything about u_c and v_c but one thing we can comment that $\frac{\partial u'}{\partial x'}$ is of the order of $\frac{\partial v'}{\partial y'}$ which means that $v_c \sim u_c\frac{h_0}{L}$, i.e. $v_c \sim \varepsilon\, u_c$ (that much we can say). So we can say that if ε is very small then v_c will be much less than u_c. This might appear to be a little bit non-intuitive at times. Let us imagine a scenario that instead of the curved boundary (as shown in figure 1(i)) there is a plate at the top which is a special case of the curved boundary. Let us assume that the plate is given a vertical velocity. So the question arises that whether the relationship $v_c \sim \varepsilon\, u_c$ still holds or not in presence of the given imposed vertical velocity. This is an interesting question which we will address later on. Even if we impose a vertical velocity the relation $v_c \sim \varepsilon\, u_c$ still holds because it has to satisfy the continuity equation $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$.

The gap height h can be a function of x and the boundary can be given a velocity. With all these considerations, it is quite obvious that it is likely to be an unsteady flow. Now we will consider the momentum equation, the x-component of the momentum equation is given by

$$\rho\left[\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right]=-\frac{\partial p'}{\partial x'}+\mu\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right)+F_x' \tag{1}$$

In equation (1), F_x' is the body force term per unit volume along the x direction. This body force term is very important because the body force itself can change the physics of the problem not the motion of the boundary. One can think of an electric field as a body force which can drive the flow and thus can alter the physics of the problem. Now we will make the scaling analysis of the various terms of equation (1). The term $\rho\frac{\partial u'}{\partial t'}$ is of the order of ~ $\rho\frac{u_c}{t_c}$, the term $\rho u'\frac{\partial u'}{\partial x'}$ is of the order of ~$\frac{\rho u_c^{\,2}}{L}$. The term $\rho v'\frac{\partial u'}{\partial y'}$ is of the order of ~ $\frac{\rho v_c u_c}{h_0}$, the term $\frac{\partial p'}{\partial x'}$ is of the order of ~ $\frac{p_c}{L}$ where p_c is the characteristic scale of pressure which is the pressure difference that is existing across x = 0 and x = L. The term $\mu\frac{\partial^2 u'}{\partial x'^2}$ is of the order of ~ $\frac{\mu u_c}{L^2}$ and the term $\mu\frac{\partial^2 u'}{\partial y'^2}$ is of the order of ~ $\frac{\mu u_c}{h_0^2}$. Let us assume that the body force term F_x' is of the order of ~ F_{xc}. This scale F_{xc} is absolutely arbitrary because we do not know about the force; it could be electrical, magnetic or whatever (it could be anything). Now we focus on the time scale t_c. In general, as we have discussed earlier, the time scale t_c can be either advective or diffusive or forcing timescale for an unsteady problem. But for a problem where there is no forced time dependence and diffusion is not that significant, it is mostly guided by the advective timescale. So, t_c becomes of the order of ~ $\frac{L}{u_c}$ and hence, both the terms

$\rho \dfrac{u_c}{t_c}$ and $\rho u' \dfrac{\partial u'}{\partial x'}$ become of the same order ~ $\dfrac{\rho u_c{}^2}{L}$. Now substituting the scale of the velocity $v_c \sim u_c \dfrac{h_0}{L}$ in the term $\rho v' \dfrac{\partial u'}{\partial y'}$, its scale also becomes of the order of ~ $\dfrac{\rho u_c{}^2}{L}$. So we find that all the three terms in the left hand side of the momentum equation (1) are essentially of the same order ~ $\dfrac{\rho u_c{}^2}{L}$. Now we introduce the dimensionless variables as $u = \dfrac{u'}{u_c}$, $v = \dfrac{v'}{v_c}$, $x = \dfrac{x'}{L}$, $y = \dfrac{y'}{h_0}$, $p = \dfrac{p'}{p_c}$ and $F_x = \dfrac{F_x'}{F_{xc}}$. Till now we have discussed about the *x*-momentum equation. But we also have the *y*-momentum equation which may also be important sometimes. Typically, if we have a vertical motion or a vertical force, then the *y*-momentum equation will be very important. This is not like a classical channel flow where the *y*-momentum equation usually gives $\dfrac{\partial p}{\partial y} = 0$. But in the present case it can give something more significant than this. Now we rewrite the *x*-momentum equation along with the dimensionless variables which is given in the following

$$\frac{\rho u_c^2}{L}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{p_c}{L}\frac{\partial p}{\partial x} + \frac{\mu u_c}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2} + F_x F_{xc} \qquad (2)$$

Now we can make some ready observations. The first observation is that the term $\dfrac{\mu u_c}{h_0^2}$ is more important than the term $\dfrac{\mu u_c}{L^2}$. This indicates the purpose of this kind of non-dimensionalization. When we non-dimensionalize, we divide a particular dimensional variable with its corresponding reference dimension. But if the reference dimension happens to be of the order of that variable itself, then the ratio comes out to be of the order of 1. For example, here we have non-dimensionalized the variable v' as $v = \dfrac{v'}{v_c}$. Since v_c is the corresponding scale of the velocity, this ratio becomes of the order of 1. But if we non-dimensionalized the variable v' as $v = \dfrac{v'}{u_c}$ that is also an acceptable form of non-dimensionalization. But this non-dimensionalization is not consistent with the scale of the problem and in that case there is no guarantee that the ratio $\frac{v'}{u_c}$ can be of the order of 1. So, here we have chosen the reference variables which are of the order of the physical scale of the problem. Once it is done, all these derivatives in equation (2) will be of the order of 1. Now to determine the importance of the individual terms of equation (2), we have to look into the coefficients that are associated with the derivatives because the derivatives are already of the order of 1. The ready conclusion that we can make is that out of the two terms $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$ and $\frac{\mu u_c}{L^2}\frac{\partial^2 u}{\partial x^2}$, $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$ will dominate because h_0 is much less than L. So in a most general physical situation we have the term $\frac{\mu u_c}{L^2}\frac{\partial^2 u}{\partial x^2}$ negligible. For example, in case of low Reynolds number hydrodynamics we know intuitively that the terms like $\rho u'\frac{\partial u'}{\partial x'}$ and $\rho v'\frac{\partial u'}{\partial y'}$ are not important because these are the advective terms. The body force term F_x' may be completely absent. Then, out of the two terms $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$ and $\frac{\mu u_c}{L^2}\frac{\partial^2 u}{\partial x^2}$, the term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$ is important. If the term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$ is important, then there has to be something which can balance this term. In the presence case, it is the pressure gradient term $\frac{p_c}{L}\frac{\partial p}{\partial x}$ which can balance this term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$. From that we can conclude about the scale of the pressure gradient term when we equate the term $\frac{p_c}{L}\frac{\partial p}{\partial x}$ with the term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$. So, we can conclude that $\frac{p_c}{L} \sim \frac{\mu u_c}{h_0^2}$, or, $p_c \sim \frac{\mu u_c L}{h_0^2} \sim \frac{\mu u_c}{\varepsilon^2 L}$ where we multiply both the numerator and the denominators by the factor L and use the definition $\varepsilon = \frac{h_0}{L}$. In every problem, whenever there is no other effect, the pressure scale is guided in this way. Of course, we can have a pressure scale which may have some dependence on the advective term. But soon we will see that the dependence of the pressure scale on the advective term will be much lower order than the dependence of the pressure scale on the viscous term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$. Once we work it out, we will realize the reason of it. So there will be other terms on which the scale of the scale of pressure can depend. But all these dependences are higher order dependences, the leading order dependence is represented by the viscous term $\frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2}$. There is a formal way to establish this. In the formal way of doing this, we take the parameter ε because ε is a small parameter.

Then we expand the variable u as $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\cdots$ (this u_0 is not the same u_0 that was drawn in figure 1(i)). This is a power series of expansion, we expand the continuous function u in a power series and this expansion can only be done if the parameter ε is small. In the similar way we can expand the variable v as $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots\cdots$ and the variable p as $p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots\cdots$. Then if we take the leading order extract, we will get the same p_c, i.e. $p_c \sim \frac{\mu u_c L}{h_0^2}$. We know that the mathematicians sometimes do a work in a certain way just to save time. Engineers and Physicists tend to do the same work in a more intuitive way with less hard work and we are trying to do the same. We can of course put $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\cdots$, $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots\cdots$ and $p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots\cdots$ in the momentum equation and then separate the variables but the leading order effect will come through the viscous term. So, at the leading order, $p_c \sim \frac{\mu u_c L}{h_0^2} \sim \frac{\mu u_c}{\varepsilon^2 L}$. From this scale we can understand that the pressure is quite strong because it scales with $\frac{1}{\varepsilon^2}$ where ε is very small. So pressure happens to be a very important consideration in the lubrication theory. Now we multiply both sides of equation (2) by $\frac{L}{p_c}$ and we get

$$\frac{\rho u_c^2}{L}\cdot\frac{L}{p_c}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\mu u_c}{L^2}\cdot\frac{L}{p_c}\frac{\partial^2 u}{\partial x^2} + \frac{\mu u_c}{h_0^2}\cdot\frac{L}{p_c}\frac{\partial^2 u}{\partial y^2} + F_x F_{xc}\cdot\frac{L}{p_c} \qquad (3)$$

By looking into all these terms of equation (3) without doing any calculation we can tell that there is one term other than the term $\frac{\partial p}{\partial x}$ which is of the order of 1. It is the viscous term $\frac{\mu u_c}{h_0^2}\cdot\frac{L}{p_c}\frac{\partial^2 u}{\partial y^2}$ because the scale of pressure p_c is chosen in such a way

($p_c \sim \dfrac{\mu u_c L}{h_0^2}$) that this term balances the pressure gradient. For the other terms we have to do a little bit of algebra. Now, $\dfrac{\rho u_c^2}{L} \cdot \dfrac{L}{p_c} \sim \dfrac{\rho u_c^2}{L} \cdot \dfrac{L h_0^2}{\mu u_c L} \sim \dfrac{\rho u_c L}{\mu}\left(\dfrac{h_0}{L}\right)^2 \sim \mathrm{Re}_L\left(\varepsilon^2\right)$ where $\dfrac{\rho u_c L}{\mu}$ is the Reynolds number based (Re_L) on L and $\dfrac{h_0}{L} = \varepsilon$. We have to keep in mind that, traditionally in lubrication theory, the Reynolds number is based on the axial length scale. The reason is that the axial length scale is usually fixed. The transverse length scale h_0' is also fixed but its physical length can vary across the length *L*. Now, $\dfrac{\mu u_c}{L^2} \cdot \dfrac{L}{p_c} \sim \dfrac{\mu u_c}{L^2} \cdot \dfrac{L h_0^2}{\mu u_c L} \sim \left(\dfrac{h_0}{L}\right)^2 \sim \varepsilon^2$; so, $\dfrac{\mu u_c}{L^2} \cdot \dfrac{L}{p_c} \sim \varepsilon^2$. Also, the body force term $F_x F_{xc} \cdot \dfrac{L}{p_c} \sim F_x F_{xc} \cdot \dfrac{L h_0^2}{\mu u_c L} \sim F_x F_{xc} \cdot \dfrac{h_0^2}{\mu u_c}$. Using the scales of the different terms, equation (3) can be rewritten as

$$\mathrm{Re}_L\, \varepsilon^2 \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + F_x F_{xc} \cdot \frac{h_0^2}{\mu u_c} \tag{4}$$

Clearly in the leading order, there is couple of terms which we can easily neglect. The terms which are of the order of ε^2, we can easily neglect them. In equation (4), all terms in the left hand side are of the order of ε^2; also the viscous term $\dfrac{\partial^2 u}{\partial x^2}$ is of the order of ε^2. Here $\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial^2 u}{\partial y^2}$ are the leading order terms, i.e. of the order of 1. But we cannot comment on the body force term because we do not know about the body force term. We cannot rule this term; this may be important or may not be important and therefore we are not ruling it out. So at the leading order we can surely rule out the term $\dfrac{\partial^2 u}{\partial x^2}$. Also we can rule out the terms in the left hand side of equation (4) provided the Reynolds number is small. If the Reynolds number is abnormally large then even with the multiplication of the small parameter ε^2 the product may become important. But in micro-scale flows, we are primarily considering the low Reynolds number hydrodynamics. So the terms which are multiplied by the product $\mathrm{Re}_L\, \varepsilon^2$ can be neglected. Then the *x*-momentum equation boils down to a simplified equation. That simplified equation is actually true only in the leading order, not for the whole term but for the leading order. The simplified form of the momentum equation is given below

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + F_x \, F_{xc} \cdot \frac{h_0^2}{\mu u_c} \tag{5}$$

So we stop at this point for the time being and we will continue from this point in the next chapter where we will look into the *y*-momentum equation. The broad objective will be to find out a distribution of the pressure between these two confining boundaries because of the complex phenomenon that is going inside. We will derive this pressure distribution in the next chapter.