Advanced Concepts In Fluid Mechanics
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Lecture – 36
Stokes Flow past a Sphere (Contd.)

In the previous chapter we have discussed about the derivation of the stream function the expression of which is given by $\psi = \frac{1}{2} u_\infty r^2 \sin^2\theta \left[1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}\right]$. This expression of the stream function for low Reynolds number flow past a sphere is a pure mathematical expression but we require this expression to calculate the drag force. We will systematically see that how one can use this expression of the stream function to calculate the drag force. The steps are very logical and we start with obtaining the expressions of the velocity components v_r and v_θ. So, using the expression of ψ, we get

$$v_r = \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{u_\infty r^2}{r^2}\left[1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}\right]\frac{1}{2\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta = u_\infty\left[1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}\right]\frac{2\sin\theta\cos\theta}{2\sin\theta}$$

$= u_\infty\left[1 - \frac{3}{2}\frac{R}{r} + \frac{R^3}{2r^3}\right]\cos\theta$ (where the factors like 2, r^2 and $\sin\theta$ get cancelled from the numerator and the denominator). Similarly, we can get the expression of v_θ, i.e.

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{2}u_\infty \frac{\sin^2\theta}{r\sin\theta}\frac{\partial}{\partial r}\left[r^2 - \frac{3}{2}Rr + \frac{R^3}{2r}\right] = -\frac{1}{2r}u_\infty \sin\theta\left[2r - \frac{3}{2}R - \frac{R^3}{2r^2}\right] =$$

$-u_\infty \sin\theta\left[1 - \frac{3}{4}\frac{R}{r} - \frac{R^3}{4r^3}\right]$ (where one $\sin\theta$ term get cancelled from the numerator and the denominator). So we have got very nice expressions of v_r and v_θ. Now we need to think of the requirement for the drag force calculation. For drag force calculation, we require the stress distribution on the sphere. The stress has two components; one is the hydrostatic component, another one is the deviatoric component. For the hydrostatic component we have to find out the pressure distribution around the sphere. Now the question arises about how one can get the pressure distribution. To get the pressure distribution, we have to substitute the expressions of the velocities v_r and v_θ in the momentum equation because in the momentum equation we have both velocity as well as pressure gradient. Let us first write the *r*-momentum equation. These forms are very cumbersome for the polar co-ordinate system; so students should never try to remember these. Although these expressions are a part of the derivation that we are doing in the class lecture, in exam if it is required then it is usually provided; so students don't need to memorize these expressions. The portions which are provided contain the momentum

equations in the polar co-ordinate system. Now the *r*-component of the momentum equation is given by

$$r\text{-momentum: } \frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2 v_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial v_\theta}{\partial \theta} + v_\theta \cot\theta\right) \quad (1)$$

These forms are so cumbersome that even if we write by seeing there can be a mistake; there is no glory in memorizing these expressions. We will just discuss about the strategy because there is no need to waste time in calculating the individual terms. We have the expressions of v_r and v_θ and for the derivatives we need to differentiate either with respect to *r* or with respect to *θ*. Substituting the expressions of the velocities v_r and v_θ along with their derivatives, the *r*-momentum equation gets remarkably simplified and we get the final form $\frac{1}{\mu}\frac{\partial p}{\partial r} = u_\infty \cos\theta \frac{3R}{r^3}$. If we integrate both sides of this equation with respect to *r*, we get $p = -\frac{3\mu u_\infty \cos\theta R}{2r^2} + \text{constant}$. In order to obtain the value of this constant, we have to apply appropriate condition. At the far stream, i.e. at $r \to \infty$, the pressure is equal to the ambient pressure p_∞, so at $r \to \infty$, $p = p_\infty$ and $\text{constant} = p_\infty$. So, the final expression of the pressure distribution becomes $p = -\frac{3\mu u_\infty \cos\theta R}{2r^2} + p_\infty$. One can have an interest to see from the θ-momentum equation whether the resulting pressure distribution is same as that obtained from the *r*-momentum equation. It should lead to the same expression of pressure; it is a cross-check just to make sure that all the calculations are done correctly. The θ-momentum equation is also equally cumbersome which is given in the following

$$\theta-\text{momentum}: -\frac{1}{\mu r}\frac{\partial p}{\partial \theta} = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \quad (2)$$

If we write the momentum equations in the vector form, it is very compact and nice, it looks very elegant. But if we expand it and do the algebra, then we have to open up the vector form (this can be compared to bringing out of a human being; for a human being there is a very beautiful outside and there is a very ugly inside. That ugly inside opens up when we open up from the vector form to get the large algebric form).

Here also we have the same strategy; we have to substitute the expressions of the velocity v_r and v_θ along with the derivatives and we get $\frac{\partial p}{\partial \theta} = -\frac{3\mu u_\infty}{2r^2} R \sin\theta$. Integrating both sides of this equation with respect to θ we get the same expression of the pressure distribution which was obtained from the *r*-momentum equation where the integral of $\sin\theta$ will be equal to - $\cos\theta$. Once we know the pressure distribution we know about the hydrostatic stress distribution. Now we need to know about the deviatoric stress. For deviatoric stress we have to apply Newton's law of viscosity. For this we need to obtain the expressions of the stress tensors τ_{rr} and $\tau_{r\theta}$. According to the polar co-ordinate system, the stress tensors τ_{rr} and $\tau_{r\theta}$ are defined as $\tau_{rr}^{Dev} = 2\mu \frac{\partial v_r}{\partial r}$ (total τ_{rr} is equal to $\tau_{rr}^{Dev} - p$) and $\tau_{r\theta} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right]$. The expression $\tau_{rr}^{Dev} = 2\mu \frac{\partial v_r}{\partial r}$ is like $\tau = \mu \frac{\partial u}{\partial y}$ expressed in a different co-ordinate system. Substituting the expression of the velocity component v_r along with its derivative we get $\tau_{rr}^{Dev} = 2\mu u_\infty \cos\theta \left[\frac{3}{2}\frac{R}{r^2} - \frac{3}{2}\frac{R^3}{r^4} \right]$. Now, the stress tensor $\tau_{r\theta}$ has only deviatoric component; it does not have any hydrostatic component because it is shear stress. Substituting the expressions of the velocity components v_r and v_θ along with their derivatives we get $\tau_{r\theta} = -\frac{3\mu u_\infty R^3 \sin\theta}{2r^4}$. So, τ_{rr} is viscous normal stress and $\tau_{r\theta}$ is viscous shear stress. As we have mentioned earlier, apart from the viscous shear stress, the viscosity can also give rise to viscous normal stress.

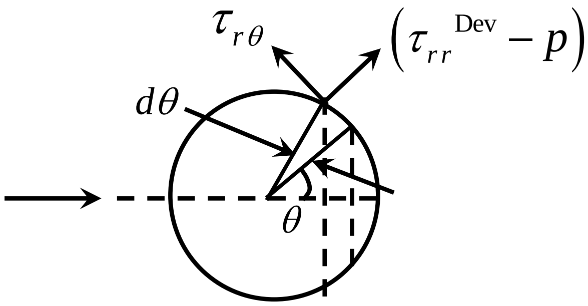

Figure 1. Stresses acting on a differential element of thickness $d\theta$ which is a part of the sphere of radius *R*. Figure is drawn in the *r*-*θ* co-ordinate system.

Now we have information about all parameters that are required for the calculation of the drag force. The drag force is clearly a function of *r* and *θ* because the stresses are functions of *r* and *θ*. To calculate the drag force, a differential element of thickness $d\theta$ is taken at an angle *θ* from the sphere of radius *R* (as shown in figure 1). The figure is drawn in the *r*-*θ* co-ordinate system where we take this element along the sphere. The normal stress that is acting on this differential element is $\tau_{rr} = \tau_{rr}{}^{Dev} - p$ while the shear stress is $\tau_{r\theta}$. Now we need to focus on the drag force which by definition is the force in the direction of the relative flow. The relative flow is in the axial direction. So we have to resolve the forces in the direction of flow. Here we need to remember that the expressions τ_{rr} and $\tau_{r\theta}$ are not forces but stresses. These stresses have to be multiplied with a suitable area to convert them to forces. The contribution of the normal stress in the drag force is given by $\left(\tau_{rr}{}^{Dev} - p\right)\cos\theta$ which needs to be multiplied by the area $2\pi R\sin\theta R d\theta$. Similarly the contribution of the shear stress in the drag force is given by $-\tau_{r\theta}\sin\theta$; again multiplied by the area $2\pi R\sin\theta R d\theta$. Here in the shear stress, negative sign is given because it is the backward direction while the flow is in the forward direction. So, the expression of the drag force is given by $F_D = \int \left[\left(\tau_{rr}{}^{Dev} - p\right)\cos\theta - \tau_{r\theta}\sin\theta\right]\Big|_{r=R} 2\pi R\sin\theta R d\theta$ where we integrate the net force acting on the differential element over the surface area of the sphere. We need to keep in mind that the expression $\left[\left(\tau_{rr}{}^{Dev} - p\right)\cos\theta - \tau_{r\theta}\sin\theta\right]$ needs to be evaluated on the sphere (i.e. at *r* = *R*) because we are calculating the drag force on the sphere. Now we have to fix the limit of the integration where the maximum value of the angle *θ* cannot be equal to 2π. The reason is that we have chosen the element within the sphere in such a way that both of the bottom half and the top half are covered. So, the value of the angle θ will lie in between 0 to π (in this way the full sphere is covered) when we evaluate the definite integral for the calculation of the drag force. Substituting the expression of the pressure $p=-\frac{3\mu u_\infty \cos\theta R}{2r^2}+p_\infty$ and limits of the angle θ, we get the modified expression of the drag force as $F_D=\int_{\theta=0}^{\theta=\pi}\left[\left(\tau_{rr}^{Dev}-p\right)\cos\theta-\tau_{r\theta}\sin\theta\right]_{r=R}2\pi R\sin\theta R\,d\theta=$

$\int_0^\pi\left[\frac{2\mu u_\infty\cos\theta}{R}(0)-p_\infty\cos\theta+\frac{3\mu u_\infty\cos^2\theta}{2R}+\frac{3\mu u_\infty\sin^2\theta}{2R}\right]2\pi R^2\sin\theta\,d\theta$. The first term is equal to zero, it does not appear. As we have already discussed in the previous chapters, ignorance can be blessing sometimes. Some people have an idea that the viscous effect does not give rise to any normal stress, so they will not consider the term τ_{rr}^{Dev}. Here luckily we have got the integral of $\int_{\theta=0}^{\theta=\pi}\left[\tau_{rr}^{Dev}\cos\theta\right]_{r=R}2\pi R\sin\theta R\,d\theta$ to be equal to zero. So even if there is any ignorance, that ignorance will not reflect in any error in the calculation of the drag force. One can easily tell the contribution of the second term $p_\infty\cos\theta$ by simply integrating it, but for better understanding one need to think it physically. This term comes from p_∞ which is a uniform pressure. Since it is a uniform pressure, if it is integrated over a spherical body the net effect will be zero. One can definitely check this upon integrating mathematically but this can be observed from the physical understanding. Then the expression of the drag force gets remarkably simplified and we have finally get the expression of the drag force as $F_D=\int_0^\pi\left[\frac{3\mu u_\infty\cos^2\theta}{2R}+\frac{3\mu u_\infty\sin^2\theta}{2R}\right]\cdot 2\pi R^2\sin\theta\,d\theta=\int_0^\pi\frac{3\mu u_\infty}{2R}\cdot 2\pi R^2\sin\theta\,d\theta$ where we have used the identity $\cos^2\theta+\sin^2\theta=1$ (so here in this derivation algebra, geometry, and trigonometry all these things that we have learnt in the school level are appearing). The final form of the drag force is given by $F_D=3\mu u_\infty\pi R\left[-\cos\theta\right]_0^\pi=6\mu u_\infty\pi R$; so, $F_D=6\pi\mu u_\infty R$ which brings the famous Stokes law. This expression gives is the force exerted by the fluid on the sphere. Now according to Newton's third law, the sphere exerts equal and opposite force on the fluid and that is how the fluid is dragged. So, one of the key issues in the Stokes law is that we have neglected the inertial term. This is one of the major assumptions otherwise it is a very classical expression. We have to check the validity of this expression. We have raised a question and we make a short discussion on the validity of the Stokes law. So the validity is dependent on the Reynolds number and instead of looking into the Reynolds number we look into the expressions of the inertia force and the viscous force. For the inertia force, we can just take one of the terms to get an idea of the order of the inertia force. The forces which are coming in the Navier-Stokes equation are all forces per unit volume. The scale of the inertia force is given by ~ $\rho v_r \frac{\partial v_\theta}{\partial r}$ (this is one of the inertial terms in the Navier-Stokes equation). If we look into the expressions of the velocity components v_r and v_θ, v_r at the leading order is equal to u_∞ while v_θ at the leading order is equal to $u_\infty \frac{R}{r}$. Since this is order of magnitude, whether there is a constant 3/2 or 3/4 appearing or not, it does not matter. When we differentiate $u_\infty \frac{R}{r}$ with respect to *r*, we get the scale as ~ $u_\infty \frac{R}{r^2}$. So, the scale of the inertia force $\rho v_r \frac{\partial v_\theta}{\partial r}$ can be written as $\rho v_r \frac{\partial v_\theta}{\partial r} \sim \rho u_\infty{}^2 \frac{R}{r^2}$ which indicates the inertia force per unit volume. Now we have to find the scale of the viscous force per unit volume which is given by ~ $\frac{\partial \tau_{rr}}{\partial r}$. This is one of the order of magnitude, we can also use $\tau_{r\theta}$ instead of τ_{rr}. If we remember the index notation in the Cartesian co-ordinate system, the stress tensor is given by $\frac{\partial \tau_{ij}}{\partial x_j}$; for the case $\frac{\partial \tau_{rr}}{\partial r}$, x_j becomes equal to *r*. When we differentiate τ_{rr} with respect to *r*, we get $\frac{\partial \tau_{rr}}{\partial r} \sim \mu u_\infty \frac{R}{r^3}$. So the ratio of the inertia force and the viscous force becomes of the order of ~ $\frac{\text{inertia}}{\text{viscous}} \sim \rho u_\infty{}^2 \frac{R}{r^2} \cdot \frac{r^3}{\mu u_\infty R} \sim \frac{\rho u_\infty R}{\mu} \cdot \frac{r}{R}$ where $\frac{\rho u_\infty R}{\mu}$ is the Reynolds number based on the radius *R* (here both the numerator and the denominator are multiplied by the factor *R* for the ease of representation). What we can conclude from this ratio is that even if the Reynolds number is small the inertia force may be much important compared to the viscous force if r is much larger than *R* which occurs far away from the sphere. One can argue that since we are calculating the drag force on the sphere then why there is a question about the importance of what is occurring at far away from the sphere. This is a wrong argument because the velocity profile which we have derived is based on the assumption that everywhere the inertial forces are zero. But if far away from the sphere, the inertial forces are non-zero, then the velocity profile will be different and therefore, the calculations of the stress tensors τ_{rr} and $\tau_{r\theta}$ will be different. So, although the calculation is on the surface of the sphere, it is on the basis of the velocity profile which is derived from the consideration of zero inertial force. This matter was first highlighted by one mathematician named Ossen who gave correction to the Stokes formula. There are many corrections to the Stokes law in the research literature. In the present course it is not very important that we have to make ourselves familiar with all those corrections. But we have to realize at least that even the Stokes law has its own limitations. So other than the major assumptions of unbounded flow and Newtonian fluid, the consideration of zero inertial force becomes questionable. If we have a situation where there is a correction in the velocity profile which depends on the inertial force; in that situation the inertial force cannot be neglected. So far we have studied mainly a collection of steady flow with low Reynolds number problems. But there are many interesting low Reynolds number problems for unsteady flows. We will look into those low Reynolds number unsteady flow problems in our subsequent chapters which are very important in microfluidic applications.