Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 35
Stokes Flow past a Sphere
(Contd.)

In the present chapter we look into the motion of a sphere in an unbounded flow with the help of a consideration in the previous chapter where we first started with a description in the Cartesian co-ordinate system and then we graduated into the polar co-ordinate system. At the end of the previous chapter we defined the two components of the velocity v_r and v_θ in terms of the velocity u_∞ at a distance far away from the sphere, i.e. $v_r = u_\infty \cos\theta$ and $v_\theta = -u_\infty \sin\theta$ at $r \to \infty$.

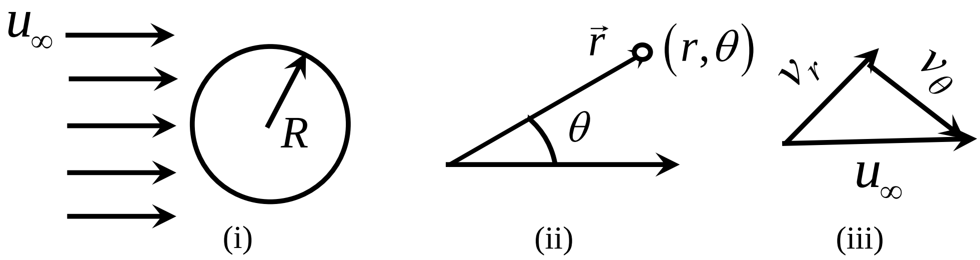

Figure 1. (i) Flow past a sphere of radius *R* where fluid is flowing with a velocity u_∞. (ii) The position of a point in a polar co-ordinate system, (iii) the components of the free stream velocity u_∞.

Now we have to obtain the Stokes stream function ψ far away from the sphere which satisfies the equation $E^2\left(E^2\psi\right)=0$. To do that we express the two components of velocity v_r and v_θ in terms of ψ, i.e. $v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$ and $v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$. These expressions are nothing but the parametric form that satisfies the incompressibility condition in the polar co-ordinate system, nothing more than that. This is true even if *r* is not tending towards infinity, but at $r \to \infty$, we have the expressions $v_r = u_\infty \cos\theta$ and $v_\theta = -u_\infty \sin\theta$. If we equate the two expressions of v_r, i.e. $v_r = u_\infty \cos\theta$ and $v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, we get $u_\infty \cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, or, $\psi = \int u_\infty r^2 \sin\theta\cos\theta\, d\theta$. If we substitute $\sin\theta = z$, then we get $\cos\theta\, d\theta = dz$ and $\int dz$ becomes equal to $\frac{1}{2}z^2$. Using this, the expression of ψ becomes $\psi = \frac{1}{2}u_\infty r^2 \sin^2\theta + f_1(r)$ where $f_1(r)$ is an integration constant. Since we are doing the partial integration with respect to θ, the constant of integration will be a function of *r* which is like a constant. I the case of partial integration, when we are integration with respect to the variable θ, any other variable is treated like a constant. Now we equate the two expressions of the velocity component v_θ, i.e. $v_\theta = -u_\infty \sin\theta$ and $v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$ from which we get $u_\infty \sin\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$. Integrating both sides with respect to *r* we get, $\psi = \int u_\infty r \sin^2\theta\, dr = \frac{1}{2}u_\infty r^2 \sin^2\theta + f_2(\theta)$. It is not surprising that the expression of ψ coming out form the integrations of the velocity profile v_r and v_θ is supposed to be same because no matter how we derive it from v_r and v_θ, the expression of ψ should be consistent. Now comparing the two results of integration of v_r and v_θ we can see that in one case there is a term function of θ while in the other case there is a term function of *r*. In order to make the expression of ψ consistent, these two functions must be equal to each other. Since one is function of *r* and the other is function of θ, they must be equal to a constant, i.e. $f_1(r) = f_2(\theta) = \text{constant}$. Otherwise the resulting expressions of ψ will not be the same. Now we need to fix the value of the constant which can be chosen arbitrarily. So for all practical purposes, we can write the expression of ψ at the far stream as $\psi = \frac{1}{2}u_\infty r^2 \sin^2\theta + \text{constant}$. The value of the constant may be set to zero as a reference. It can be any constant not necessarily zero; but for algebraic convenience we can set it to zero. So we have got the information for ψ at the far stream at $r \to \infty$. Therefore, the question arises about what will be the value of ψ if *r* is not tending towards infinity. Typically *r* is small if it is close to the surface of the sphere. As we go from the surface of the sphere to far away from the sphere, the *r* dependence changes. Because it is the radial co-ordinate *r* that demarcates whether it is in the vicinity of the sphere or it is far away from the sphere. The θ-dependence can be very similar whether it is close to the surface of the sphere or far away from the sphere. So we can write that for a finite *r*, the θ-dependence is $\sin^2\theta$ and $\psi = f(r)\sin^2\theta$. The value of the function

$f(r)$ is equal to $\frac{1}{2}u_\infty r^2$ if r is tending towards infinity, i.e. $f(r)=\frac{1}{2}u_\infty r^2$ at $r\to\infty$. But the value of the function $f(r)$ will be something else for a finite *r*. So this form $f(r)\sin^2\theta$ is a separation of variables essentially where the one part is a function of *r* only while the other part is a function of θ only. If one looks into the mathematics book, they will start the solution from this particular step. But there should be a physical basis of why we are choosing this particular form and that is very important for our conceptual understanding. So the reason of choosing this particular form is clarified. Now we calculate the operator E^2. From definition, the E^2 operator is given by $E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right]$. Since, ψ is given by the expression $\psi=f(r)\sin^2\theta$, $E^2\psi=f''\sin^2\theta+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}f\cdot 2\sin\theta\cos\theta\right]$. The $\sin\theta$ term gets canceled from the numerator and the denominator and we get $E^2\psi=\left(f''-\frac{2f}{r^2}\right)\sin^2\theta$. $f''-\frac{2f}{r^2}$ is a function of *r*, let us define it as a function $g(r)$ so that $g(r)=f''-\frac{2f}{r^2}$ (since f is a function of *r*, its derivative and $\frac{f}{r^2}$ will also be function of *r*); so, $E^2\psi=g(r)\sin^2\theta$. In the similar fashion, $E^2\left(E^2\psi\right)$ will be eqal to $\left(g''-\frac{2g}{r^2}\right)\sin^2\theta$ (this is like an aptitude test), i.e. $E^2\left(E^2\psi\right)=\left(g''-\frac{2g}{r^2}\right)\sin^2\theta$ where we just replace *f* with *g*. So the equation $E^2\left(E^2\psi\right)=0$ essentially boils down to

$$g''-\frac{2g}{r^2}=0 \tag{1}$$

This equation has a solution where *g* is of the form of r^n. In fact the function f is in the form of r^n, we start with the function *f* and we move into the function *g*. So both *f* and *g* are of the form of r^n (where the value of *n* needs to be determined). Let $f=Ar^n$, so, the first derivative $f'=Anr^{n-1}$ and the second derivative $f''=An(n-1)r^{n-2}$ and $\frac{2f}{r^2}=2A\,r^{n-2}$. So, $g=f''-\frac{2f}{r^2}=A\left[n(n-1)-2\right]r^{n-2}$; let us assume $A\left[n(n-1)-2\right]$ to be equal to *B*, i.e. $A\left[n(n-1)-2\right]=B$. Now we evaluate the expression $g''-\frac{2g}{r^2}$ which is equal to $g''-\frac{2g}{r^2}=B\left[(n-2)(n-3)-2\right]r^{n-4}$ where the parameter *n* is replaced by *n*-2 (just like aptitude test). Since, $A\left[n(n-1)-2\right]=B$, the expression of $g''-\frac{2g}{r^2}$ becomes $g''-\frac{2g}{r^2}=A\left[n(n-1)-2\right]\left[(n-2)(n-3)-2\right]r^{n-4}$. So, $g''-\frac{2g}{r^2}=0$ means that $\left[n(n-1)-2\right]\left[(n-2)(n-3)-2\right]=0$. This is a multiplication of two quadratic expressions of *n* which means that there will be four roots. Now expanding one of the two terms $n(n-1)-2$ we get the result n^2-n-2 which can be written as $n^2-n-2=n^2-2n+n-2=n(n-2)+1(n-2)=(n+1)(n-2)$; so the expression n^2-n-2 is factorized into the products of $(n+1)$ and $(n-2)$. Similarly, we expand the other term $(n-2)(n-3)-2$ to get the result n^2-5n+4 which can be written as $n^2-5n+4=n^2-4n-n+4=n(n-4)-1(n-4)=(n-1)(n-4)$; so the expression n^2-5n+4 is factorized into the products of $(n-1)$ and $(n-4)$. In the school level (may be in class 7 or class 8) we learnt about the factorization; at that time we could not realize that for a glorified thing like the Stokes flow past a sphere this factorization can be required (but in reality we require this). So, the expression $\left[n(n-1)-2\right]\left[(n-2)(n-3)-2\right]=0$ is now rewritten as $(n+1)(n-2)(n-1)(n-4)=0$. So, *n* is equal to $n=-1, 1, 2$ and 4. We use the general form of *f* , i.e. $f=c\,r^n$. Because of the linearity of the solutions for all values of *n*, they should be linearly superimposed to get the general solution. So, the general solution using all values of *n* is given by $f=c_1 r+c_2 r^2+c_3 r^4+\frac{c_4}{r}$. To obtain the constants c_1, c_2, c_3 and c_4 we need to apply the appropriate boundary conditions. In order to understand the boundary conditions, let us consider figure 1 (i) where shows the flow past a sphere of radius *R*. Before writing any boundary condition mathematically it is very important to understand the physical picture of the boundary condition. Then only we will never make a mistake of writing the boundary conditions. At the surface of the sphere there are two important considerations. One of them is a pure kinematic consideration that fluid cannot penetrate through the sphere which is called as no-penetration boundary condition. So, at $r=R$,

$v_r = 0$; first we will write in terms of velocity and then we will convert it in terms of the stream function. Also, at $r = R$, $v_\theta = 0$ because of the no-slip boundary condition. Now, at $r \to \infty$ the stream function should be such that it become equal to what we have derived earlier, i.e. at $r \to \infty$, $\psi = \frac{1}{2} u_\infty r^2 \sin^2\theta$. In fluid mechanics, there is a language (like in everything there is a grammar); this language $r \to \infty$, $\psi = \frac{1}{2} u_\infty r^2 \sin^2\theta$ is called as the far stream boundary condition. Sometimes people do not write it explicitly, but far stream boundary condition means it should match with what happens at the far stream (whatever solution that we are coming up, it should match with that at the far stream). Now what is left is to convert the boundary conditions of velocity in terms of the stream function. No penetration boundary condition means that the flow is always tangent to the surface of the sphere but there is no flow normal to the surface of the boundary. This is nothing but the definition of a streamline where we always have a flow tangent to the streamline but there is no flow normal to the streamline. Since there is no flow normal to the surface, this acts as a streamline which by definition is having a constant value of stream function. So ψ is equal to a constant over the surface of the sphere; we can take the value of the constant arbitrarily but we take it as zero for simplicity. So, this no-penetration boundary condition becomes equivalent to $f = 0$ at $r = R$. In the Cartesian co-ordinate system, the velocity u is defined in terms of the derivative of ψ with respect to *y* and here in the polar co-ordinate system, v_θ is defined as the derivative of ψ with respect to *r*. The derivative of ψ with respect to *r* is nothing but the implication of $f' = 0$. So the no-slip boundary condition results $f' = 0$ at $r = R$. So the three boundary conditions are at (i) $r = R$, $f = 0$; (ii) at $r = R$, $f' = 0$ and (iii) at $r \to \infty$, $\psi = f \sin^2\theta$ where $f = \frac{1}{2} u_\infty r^2$. The interesting part of this mathematical problem is that we have four constants c_1, c_2, c_3 and c_4 respectively but we have three boundary conditions. Using the physical nature of the problem, we can eliminate one of these constants. Now let us put these constraints, at $r = R$, $f = 0$ means $c_1 R + c_2 R^2 + c_3 R^4 + \frac{c_4}{R} = 0$. Also at $r = R$, we have $f' = 0$ which means $c_1 + 2c_2 R + 4c_3 R^3 - \frac{c_4}{R^2} = 0$ and at $r \to \infty$, we have

$f=\frac{1}{2}u_\infty r^2$. From the condition $r\to\infty$, $f=\frac{1}{2}u_\infty r^2$, the first thing that we can conclude is the value of the constant c_2 which is $c_2=\frac{u_\infty}{2}$ because the r^2-dependence in the function *f* should match with the r^2-dependence in the boundary condition. We not only get the constant c_2 but also we can say that $c_3=0$ because the maximum dependence on *r* at $r\to\infty$ cannot go beyond r^2 (i.e. the maximum value of power of *r* cannot go beyond 2); so, the value of the constant c_3 must be equal to zero. Once we have obtained the value of c_3, i.e. $c_3=0$, the expressions $c_1 R+c_2 R^2+c_3 R^4+\frac{c_4}{R}=0$ and $c_1+2c_2 R+4c_3 R^3-\frac{c_4}{R^2}=0$ become simplified to $c_1 R+c_2 R^2+\frac{c_4}{R}=0$ and $c_1+2c_2 R-\frac{c_4}{R^2}=0$ respectively and we can find the remaining constants c_1 and c_4. Now we divide both sides of the equation $c_1 R+c_2 R^2+\frac{c_4}{R}=0$ by *R* to get $c_1+c_2 R+\frac{c_4}{R^2}=0$. If we add the two equations $c_1+c_2 R+\frac{c_4}{R^2}=0$ and $c_1+2c_2 R-\frac{c_4}{R^2}=0$, we get $2c_1+3c_2R=0$. Substituting the value of $c_2=\frac{u_\infty}{2}$, we get $c_1=-\frac{3u_\infty R}{4}$. Substituting the values of c_1 and c_2 in the equation $c_1+c_2 R+\frac{c_4}{R^2}=0$ we get $\frac{c_4}{R^2}=-c_1-c_2 R=c_1=\frac{3u_\infty R}{4}-\frac{u_\infty R}{2}=\frac{u_\infty R}{4}$. So the procedure is pretty straightforward; one can obviously go through the algebric details to check whether everything is done properly or not. The final expression of ψ is given by $\psi=\frac{1}{2}u_\infty r^2\sin^2\theta\left[1-\frac{3}{2}\frac{R}{r}+\frac{R^3}{2r^3}\right]$.

In principle, the stream function is everything because once we know the expression of the stream function of the problem we can obtain the velocity distribution since we have the expressions for velocity as a function of the stream function. Now the end objective will be to calculate the drag force on the sphere. In our next chapter we will see that starting from the expression of the stream function how one can calculate the drag force on the sphere. But we need to remember that the stream function ψ is the most fundamental thing because it gives the velocity field. In fluid mechanics, if we have the velocity field we can get all other derived parameters from the velocity field itself and that part will be taken up in the next chapter.