Advanced Concepts In Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 34
Stokes Flow past a Sphere

In the previous chapters we have discussed about low Reynolds number flows through geometries like channels and pipes. Now there are also other types of low Reynolds number flows which are very interesting. But they are the kind of flows which do not take place within a confinement but flow across a particular body, for example, flow past a spherical body.

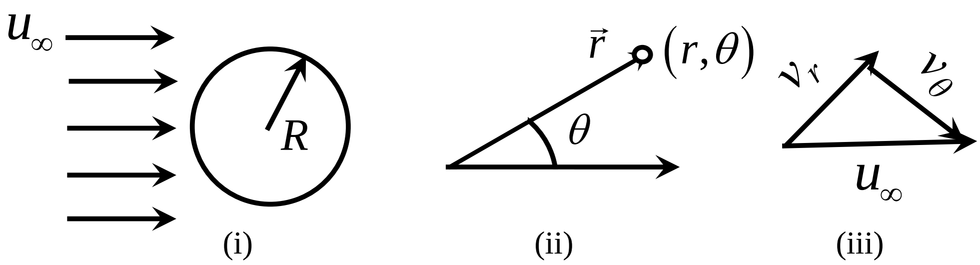

Figure 1. (i) Flow past a sphere of radius R where fluid is flowing with a velocity u_∞. (ii) The position of a point in a polar co-ordinate system, (iii) the components of the free stream velocity u_∞.

The classical example is that, we have a sphere and we have a fluid which flows past the sphere. To make an analysis in this case, we assume that there is a sphere of radius R and the fluid is flowing past it. This is very classical to one experiment which is known as the Stokes experiment. If we recall from our high school physics, there is an experiment of the falling sphere. There is a sphere which is falling in viscous oil and when the sphere is falling in viscous oil, the weight of the sphere is acting downwards, the buoyancy force is acting upwards and the viscous drag force is acting upwards against the direction of the motion. Under the combined action of these three forces, the sphere comes to equilibrium and the corresponding velocity is called as the terminal velocity. Experimentally it is very interesting to measure the terminal velocity and compare the terminal velocity with a mathematical formula which is also known as the Stokes law. So the objective of whatever we are going to do in the present chapter is to derive the Stokes law. In the high school Physics the Stokes law was considered to be a magical formula; here in the present chapter we will go step by step to derive this. It is one of the very involved derivations in fluid mechanics, but at the same time very classical. We need to go through very carefully. The challenge here is the spherical nature of the geometry. It is one of the challenges; but there can be several other challenges. So, here, one of the challenges is the spherical nature of the geometry.

To keep the formulation to be geometry independent at least for the initial few steps, we will use the vector form of the Navier-Stokes equation. When we are writing the Navier-Stokes equation, the first assumption that we are making is that the fluid which is flowing around the sphere is homogeneous, isotropic, Newtonian fluid. When the sphere is stationary and the fluid is flowing past it, it is same as the fluid being stationary and sphere moving past it because it is all about the relative velocity. This is very common in aerospace engineering. In aerospace engineering, when an aircraft moves in order to calculate the force on the aircraft people make wind tunnel experiments where they fix the aircraft and make the wind flowing past it. So it is the relative velocity that is important not really the absolute velocity. So, in that respect, in reality the fluid is may be stationary and the sphere is moving, but here we have kept the sphere stationary and made the fluid moving. The velocity with which the fluid is coming from the far stream is u_∞. We will start with the Navier Stokes equation in the vector form which is $0=-\nabla p+\mu\nabla^2\vec{v}$. Sometimes gravity as a body force can be included in the pressure gradient term and this is very common thing that we do. The reason is that we can always redefine the pressure p as $p+\rho g h$ which is called as the piezometric pressure. It is equal to the summation of the pressure and the pressure equivalent of the gravity head. So either way we can absorb the gravity body force term in the pressure gradient term. Here, of course, we are not much concerned about the gravity force; we are just interested in the drag force. As already stated, we have considered this flow as the low Reynolds number flow; this is one of the key assumptions. So we are neglecting the inertial terms. We will later on question ourselves that trivially for all low Reynolds number flows, whether the inertial term can be set equal to zero or not. Since Reynolds number by definition is the ratio of the inertia force and the viscous force, for low Reynolds number we can neglect the inertia force; that is the intuitive way of looking into it. As we mentioned earlier, one of the challenges in dealing with the equation $0=-\nabla p+\mu\nabla^2\vec{v}$ is that we have a pressure term as an unknown. But we do not have a separate governing equation for the pressure. So, one of the key strategies is to eliminate pressure from this equation. In order to eliminate pressure we need to take curl (i.e. using the curl operator) on both sides of the equation $0 = -\nabla p + \mu \nabla^2 \vec{v}$. Since curl of gradient of a scalar is a null vector, taking the curl operator on the pressure gradient term ∇p leads to a null vector and we get $0 = \mu \nabla \times \left(\nabla^2 \vec{v}\right)$. Now, for further simplification, we use the

vector identity $\nabla \times \left(\nabla \times \vec{v}\right) = \nabla\left(\nabla \cdot \vec{v}\right) - \nabla^2 \vec{v}$. This is such a common vector identity that it is inevitably used in fluid mechanics. In this context it is extremely important to highlight that it is not a special skill to memorize this vector identity. People like Professors can reproduce it easily from their years of experience in teaching, but it is not at all important to memorize this. It is however important to make use of this identity to develop a good insight; that is more important. So, one major observation from this vector identity is that if it is an incompressible flow, then $\nabla \cdot \vec{v}$ becomes equal to zero since for incompressible flow, the divergence of the velocity vector is equal to zero. Now the curl of the velocity vector $\nabla \times \vec{v}$ in fluid mechanics is called as the vorticity vector $\vec{\Omega}$ and the vector identity becomes $\nabla \times \vec{\Omega} = -\nabla^2 \vec{v}$. Now the vorticity vector $\vec{\Omega}$ physically represents the rotation in the flow. So, the rotational effect in the flow becomes related to the viscous effect in the flow since the term $\mu \nabla^2 \vec{v}$ represents the viscous force term in the Navier-Stokes equation. So, very interestingly we can see that how mathematics can represent physics. $\nabla \times \left(\nabla \times \vec{v}\right) = -\nabla^2 \vec{v}$ is a vector calculus formula which does not understand any physics but it nicely represents the physics by relating the curl of the vorticity vector with the viscous force. Now question arises about how this has been possible. Physically, in school level language, it means that we have the rotational effect because of viscous interactions. To understand this one can think of the following example. Let us consider that a bus is moving and we have just come out or jumped out of the moving bus (or moving car). If we have jumped out of the bus, we will see that until and unless we maintain our motion further forward till some time, we will have a tendency to topple. The reason is that when we are in the bus, we are having an inertia and when we are touching the ground that inertia is disturbed because of the frictional resistance. In case of fluid flow, viscosity creates the frictional resistance. Because of that frictional resistance, there is a rotational effect that is created within it and that makes the tendency of toppling. In this way viscosity induces rotationality in the flow (at least curl of the rotationality or vorticity). Using the simplified form of the vector identity $\nabla\times\vec{\Omega}=-\nabla^2\vec{v}$, the governing equation becomes $0=\nabla\times\left(\nabla\times\vec{\Omega}\right)$ where we divide both sides of the equation by μ since the viscosity μ is non-zero. So the major assumption from the step $0=\mu\nabla\times\left(\nabla^2\vec{v}\right)$ to the step $0=\nabla\times\left(\nabla\times\vec{\Omega}\right)$ is the assumption of an incompressible flow. So, overall, it is a low Reynolds number flow as well as the incompressible flow. Additionally there is consideration like Newtonian fluid (and assumptions like homogeneous, isotropic fluid) which is not rewritten here because writing the Navier Stokes equation itself implies that we are indeed considering the Newtonian fluid. Now we use the same vector identity again where we just replace the velocity vector $\vec{v}$ with the vorticity vector $\vec{\Omega}$ and we get $\vec{0}=\nabla\left(\nabla\cdot\vec{\Omega}\right)-\nabla^2\vec{\Omega}$. If we substitute the expression of the vorticity vector $\vec{\Omega}=\nabla\times\vec{v}$ in $\vec{0}=\nabla\left(\nabla\cdot\vec{\Omega}\right)-\nabla^2\vec{\Omega}$, we get $0=-\nabla^2\left(\nabla\times\vec{v}\right)$ because the divergence of curl of any vector is equal to null vector which is another vector identity. So, our governing equation boils down to $\nabla^2\vec{\Omega}=0$. Now let us take a different example, i.e. an example of a two-dimensional flow while our present problem is a three-dimensional flow where the flow may take place in a plane but the geometry is three-dimensional. But we now focus on the two-dimensional problem where we have only two components of velocity u and v. In this context, it is important to note that the present problem can also be boiled down to a two-dimensional flow because if we see the flow around the sphere, it can be perceived or simulated by the flow through a central plane. There the velocity components will be v_r and v_θ; there will be no v_z component. Although the geometry of a sphere is a three-dimensional geometry, the flow past a sphere can be nicely modeled as a two-dimensional flow. But the difference is that this two-dimensional approximation of flow past sphere is not in the Cartesian co-ordinates. It may be described in a Cartesian co-ordinate system, but that is not most conveniently described in a Cartesian system. It is most commonly described in a spherical polar co-ordinate system. So here we start with the example of a two-dimensional flow described in a Cartesian co-ordinate system. In a Cartesian co-ordinate system, the angular velocity $\left(\vec{\omega}_z\right)$ with respect to the *z* axis (where we have flow in the *x*-*y* plane) is given by $\vec{\omega}_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\hat{k}$. The vorticity $\left(\vec{\Omega}\right)$ is the double of the angular velocity, i.e. $\vec{\Omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$. The expression $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ quantifies the rate of deformation and the expression $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ quantifies the rate of rotation. So, the rate of rotation can be represented by either the angular velocity vector $\vec{\omega}_z$ or the vorticity vector $\vec{\Omega}$ (just a difference of a factor 1/2).

Now to solve our problem to obtain the velocity profile, there is still challenge remaining because in the expression of $\vec{\Omega}$, we still have two components u and v. In order to have a differential equation comfortable, no matter how complex the differential equation is, we want a single dependent variable. Independent variables may be many but dependent variable should be single. But in the present case there are two dependent variables u and v. We can convert these two variables into a single independent variable by parametrizing u and v; i.e. writing the parametric forms of u and v. That parametric form can be written by appealing to the continuity equation. For two-dimensional incompressible flow we have $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. We define $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$; so ψ is a function of both x and y which is called as stream function. So, the objective of introducing this function ψ is to setup the parametric form of the incompressibility constraint $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, nothing more than that. If we define the two variables u and v in this particular way, then we can see that the equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is automatically satisfied. Whether u is positive or negative and v is negative or positive, the cases are equivalent because the summation of these two has to be equal to zero. So, we can also define u as $u = -\frac{\partial \psi}{\partial y}$ and v as $v = \frac{\partial \psi}{\partial x}$. Using this definition in the expression of the vorticity vector $\vec{\Omega}$ we get $\vec{\Omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \hat{k}$. The expression $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$ in two-dimension is the Laplacian of the variable ψ, i.e. $\vec{\Omega} = -\nabla^2 \psi \, \hat{k}$. So, the equation $\nabla^2 \vec{\Omega} = 0$ boils down to $\nabla^2 \left(\nabla^2 \psi \right) = 0$ which becomes our governing equation in the Cartesian reference frame. In the polar co-ordinate system, we cannot use exactly this from because of the changes in the geometry; we are no more using the *x-y-z* type of co-ordinate system. So in the polar co-ordinate system, we replace the Laplacian operator ∇^2 with the new operator E^2 which satisfies the equation $E^2\left(E^2\psi\right)=0$. This ψ is not defined in the same way as the Cartesian, because now the stream function has to satisfy the polar co-ordinate version of the continuity equation, not the Cartesian co-ordinate version. So this ψ is a different stream function which is called as the Stokes stream function. Now we will write the expression of the operator E^2 which is given by $E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right]$; we will use the operator E^2 later on. To understand about r and θ, we consider a plane (very similar to the polar co-ordinate system) where there is polar angle θ and the radial direction $\vec{r}$. A point in this system will have a co-ordinate $\left(r,\theta\right)$ as shown in figure 1 (ii). Now one interesting thing is that once we know the value of ψ or the expression of ψ at the far stream we can use that as a guess to obtain ψ for the scenario when we come close to the sphere. To do this let us first understand about the flow far away from the sphere. Far away from the sphere means that the influence of the sphere is gone and it is again parallel like the free stream velocity u_∞; so it will be u_∞ horizontal. This u_∞ will have its own components v_r and v_θ (as shown in figure 1 (iii)). The angle between v_r and v_θ is 90^0; we have vectorically resolved into two vectors v_r and v_θ. So, $v_r=u_\infty\cos\theta$ and $v_\theta=-u_\infty\sin\theta$; the minus sign is given because it is the opposite direction of θ. These are to conditions to be valid at $r\to\infty$. If we measure the radial co-ordinate from the centre of the sphere at $r\to\infty$, i.e. at far away from the sphere we get $v_r=u_\infty\cos\theta$ and $v_\theta=-u_\infty\sin\theta$. Now the question arises about how this can lead to an expression of ψ at $r\to\infty$. This is very important and we will take it up from this thing in the next chapter. It is very much important because based on this we will consider a form of ψ in the equation $E^2\left(E^2\psi\right)=0$ and we will try to satisfy that particular form; we will take that up in the next chapter.