Advanced Concepts In Fluid Mechanics
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Lecture – 33
Potential Flow (Contd.)

In the previous chapter we have discussed about a general theorem which can enable us to calculate the lift force and the drag force because of a potential flow around a closed contour. That theorem is known as the Blasius force theorem. In the present chapter, we will try to derive the Blasius force theorem and we will see that by considering the special case of flow past a circular cylinder with rotation how the Blasius force theorem can be used to calculate the lift force and the drag force.

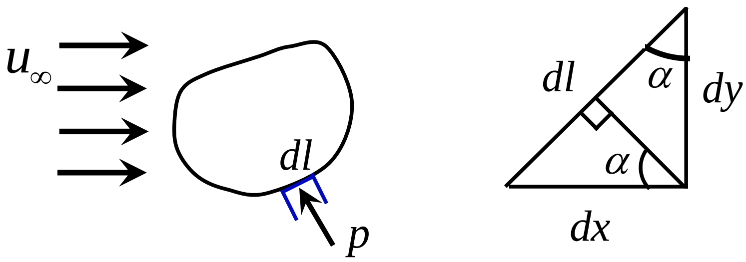

Figure 1. Flow past a body of arbitrary shape where u_∞ is the far stream velocity. '*dl*' is differential small element on which the pressure *p* is acting. The element '*dl*' is resolved into two components '*dx*' and '*dy*' on the right hand side of the figure.

Let us consider that we have an arbitrary shaped body and we have a potential flow past this body. We can take a small element '*dl*' and on that small element we have a pressure *p* which is perpendicular to the element. Let us say that this pressure *p* makes an angle *α* with the horizontal line along which this incipient flow is taking place. The velocity u_∞ is coming from the far stream. Now if we consider the component '*dl*', then it will have a component '*dx*' along the *x* direction and component '*dy*' along the *y* direction. The direction of the pressure *p* is such that it makes an angle *α* with '*dx*'. The horizontal component of the force is dF_D and the vertical component of the force is dF_L. The horizontal component is the drag force and the vertical component is the lift force. The horizontal component of the force corresponding to pressure is given by $dF_D = -p\cos\alpha\, dl$. Because of the direction of the pressure, the horizontal component is in the opposite direction of u_∞ and therefore a minus sign has been given in the expression of dF_D. Similarly, the vertical component is given by $dF_L = p\sin\alpha\, dl$. From the figure one can understand that $\cos\alpha\, dl$ is nothing but '*dy*' and $\sin\alpha\, dl$ is nothing but '*dx*'. So we get $dF_D = -p\,dy$ and $dF_L = p\,dx$ from which we can calculate $dF_D - i\,dF_L$ as $dF_D - i\,dF_L = -p\,dy - i\,p\,dx = -i\,p\left(dx - i\ dy\right)$ using $i^2 = -1$. We form this complex quantity $dF_D - i\,dF_L$ in which we absorb both the drag force and the lift force. Now $dx - i\ dy$ is the complex conjugate of *dz* = *x* + *i* *y* which we define as *dz**, so, $dz^* = dx - i\ dy$. We need to remember that we have taken an arbitrary shaped body, not necessarily a circular cylinder. So the contour integral of $dF_D - i\,dF_L$ can be written as $\oint\left(dF_D - i\,dF_L\right) = -\oint i\,p\left(dx - i\ dy\right)$. Now it is noteworthy that all the assumptions of flow past circular cylinder remain valid here also and we can write $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho V^2$. Here, V^2 is nothing but equal to $u^2 + v^2$; so, $V^2 = u^2 + v^2 = \left(u + i\,v\right)\left(u - i\,v\right)$. We will use the expression of *p* from the equation $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho V^2$ and then we will substitute it in the contour integral $-\oint i\,p\left(dx - i\ dy\right)$. We get, $-\oint i\,p\left(dx - i\ dy\right) = -\oint i\left(p_\infty + \frac{1}{2}\rho u_\infty{}^2 - \frac{1}{2}\rho\left(u + i\,v\right)\left(u - i\,v\right)\right)\left(dx - i\ dy\right)$. One obvious thing in this integral is that integral of $p_\infty + \frac{1}{2}\rho u_\infty{}^2$ (which is a constant) over a closed contour is equal to zero. It can be physically felt that if we have a uniform pressure distribution around a closed body, the net force because of the uniform pressure distribution is equal to zero. So the $p_\infty + \frac{1}{2}\rho u_\infty{}^2$ part will not contribute to the integration and the contour integral $-\oint i\,p\left(dx - i\ dy\right)$ becomes $-\oint i\,p\left(dx - i\ dy\right) = \frac{i\,\rho}{2}\oint\left(u + i\,v\right)\left(u - i\,v\right)\left(dx - i\ dy\right)$. Now we calculate the expression of $\left(u + i\,v\right)\left(dx - i\ dy\right)$. $\left(u + i\,v\right)\left(dx - i\ dy\right) = u\,dx - u\,i\,dy + v\,i\,dx + v\,dy$ using $i^2 = -1$, so, it becomes $\left(u + i\,v\right)\left(dx - i\ dy\right) = u\,dx + v\,dy + i\left(v\,dx - u\,dy\right)$. Now question arises about whether we can tell the corresponding expression of $\left(u + i\,v\right)\left(dx - i\ dy\right)$ on the surface of the body. The surface of the body is a streamline and streamline is defined by the equation $\frac{dx}{u} = \frac{dy}{v}$. Therefore, $v\,dx - u\,dy = 0$ on the surface of the body and we get

$(u+iv)(dx-i\,dy)=u\,dx+v\,dy$. Just for the sake of interest, we calculate the expression $(u-iv)(dx+i\,dy)$, which is given by $(u-iv)(dx+i\,dy)=u\,dx+u\,i\,dy-v\,i\,dx+v\,dy$, or, $(u-iv)(dx+i\,dy)=u\,dx+v\,dy+i(u\,dy-v\,dx)$. $u\,dy-v\,dx$ is equal to zero on the surface and therefore, $(u-iv)(dx+i\,dy)=u\,dx+v\,dy$. Therefore, on a streamline, the expressions $(u+iv)(dx-i\,dy)$ and $(u-iv)(dx+i\,dy)$ are the same. Hence, we replace $(u+iv)(dx-i\,dy)$ of the contour integral with $(u-iv)(dx+i\,dy)$ and the contour integral becomes $-\oint i\,p(dx-i\,dy)=\frac{i\rho}{2}\oint(u-iv)^2(dx+i\,dy)$. Therefore, the net result is the following expression $F_D-i\,F_L=\frac{i\rho}{2}\oint(u-iv)^2(dx+i\,dy)=\frac{i\rho}{2}\oint\left(\frac{dF}{dz}\right)^2 dz$. Here we have substituted $\frac{dF}{dz}=u-iv$ (using the definition of the complex potential function) and $dz=dx+i\,dy$. So, if we know the complex potential function, just by using the derivative of the complex potential function we can calculate the lift force and the drag force immediately. This highlights the power of this complex analysis. So, $F_D-i\,F_L=\frac{i\rho}{2}\oint\left(\frac{dF}{dz}\right)^2 dz$ is called as the Blasius force theorem. Now let us illustrate the use of this theorem through the example of flow past a circular cylinder with circulation or with rotation.

In this case, the complex potential function F is given by $F=u_\infty z+\frac{u_\infty R^2}{z}-\frac{i\Gamma}{2\pi}\ln z+(a+ib)$, then $\frac{dF}{dz}=u_\infty-\frac{u_\infty R^2}{z^2}-\frac{i\Gamma}{2\pi z}$. From this we get, $\left(\frac{dF}{dz}\right)^2=u_\infty{}^2+\frac{u_\infty{}^2R^4}{z^4}+\frac{i^2\Gamma^2}{4\pi^2z^2}-\frac{2u_\infty{}^2R^2}{z^2}-\frac{2u_\infty i\Gamma}{2\pi z}+\frac{2u_\infty R^2 i\Gamma}{2\pi z^3}$ by using the expansion of $(a+b+c)^2$. So we can see that $\left(\frac{dF}{dz}\right)^2$ is a complex number which can be represented as a power series of z. In general, this is the scenario for a function which is an arbitrary complex function. Now let us write the function F.

We start from the extreme negative coefficient to move towards the extreme positive coefficients and we write only the intermediate coefficients as

$f(z) = ...c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1} + c_0(z-a)^0 + c_1(z-a)^1 + c_2(z-a)^2 + ...$. In this way it does from one extreme on the left hand side to the other extreme on the right hand side. This is also called as Laurent series which is a way of writing a series expansion of a complex function. So the present example is very similar to this Laurent series expansion with $a=0$. Interestingly, the function in the Laurent series is singular at $z=a$. $z=a$ is called as a point of singularity, because in the denominator of the terms of this expansion, there will be term like $z-a$ and at $z=a$, it will be of the form of $\frac{1}{0}$. So, in the general form of expansion in the Laurent series, $z=a$ is the point of singularity and in the present example $a=0$ is the point of singularity. Let us think of calculating the integral $\oint \frac{dz}{z}$. We have $z = r\,e^{i\theta}$; we are calculating this integral on the surface of the cylinder, so $r=R$ and we are varying θ. So, $z = Re^{i\theta}$ and $dz = Ri\,e^{i\theta}d\theta$ and the integral becomes $\oint \frac{dz}{z} = \int_0^{2\pi} \frac{Ri\,e^{i\theta}d\theta}{Re^{i\theta}} = 2\pi i$. So we can generalize this by saying that if there is a coefficient c_{-1}, then the integral will be equal to $2\pi c_{-1}$. This is called as Residue theorem. Although this is considered as a theorem, it is very intuitive and easy to derive. Using this theorem, the corresponding integrals of all other coefficients will be equal to zero. We can show it very easily. If we calculate the integral $\int \frac{dz}{z^n}$, we will see that except for the value of $n=1$, all other integrals will be non-existent. So, what will matter is the coefficient of $\frac{1}{z}$. The coefficient of $\frac{1}{z}$ in the expression of $\left(\frac{dF}{dz}\right)^2 = u_\infty{}^2 + \frac{u_\infty{}^2R^4}{z^4} + \frac{i^2\,\Gamma^2}{4\pi^2\,z^2} - \frac{2u_\infty{}^2R^2}{z^2} - \frac{2u_\infty\,i\,\Gamma}{2\pi\,z} + \frac{2u_\infty R^2\,i\,\Gamma}{2\pi\,z^3}$ is $-\frac{2u_\infty\,i\,\Gamma}{2\pi}$. We now recall the expression of $F_D - i\,F_L$ which is given by $F_D - i\,F_L = \frac{i\,\rho}{2}\oint\left(\frac{dF}{dz}\right)^2 dz$. Now we will evaluate the integral using the aforesaid theorem to get $F_D - i\,F_L = \frac{i\,\rho}{2} \times 2\pi i\left(-\frac{2u_\infty\,i\,\Gamma}{2\pi}\right) = i\,\rho\,\Gamma\,u_\infty$. If we compare the real part and the imaginary part from the both sides we will see that the drag force is still zero but the lift force is equal to $-\rho\,\Gamma\,u_\infty$. This is known as Kutta-Zhukhovski theorem.

So $F_L = -\rho \Gamma u_\infty$ is the lift force. We can see very nicely that, up to the step $F_D - i F_L = \dfrac{i\rho}{2} \oint \left(\dfrac{dF}{dz} \right)^2 dz$, we can calculate the lift force on any shape. The present example is of course a special case of flow past a circular cylinder but it can give us a very nice overview of the physics associated in the calculation of the lift force. We can clearly see that if Γ is anti-clockwise, then F_L is negative and if Γ is negative (i.e. if Γ is clockwise), then F_L is positive. It means that if there is an object which is rotating in a clockwise fashion it will experience a lift force upwards. If it is rotating in an anticlockwise direction, it will be a negative lift force; it will be experiencing a force downwards. So we have understood, at least in principle, the procedure to model the flow past an object of a circular shape and the resulting calculation of the lift force and the drag force.

Now the question is that, in reality, all the objects are not circular or cylindrical. So, there can be situations when we have an object which is of a different shape. In that case question arises about how we can use the result of the flow past a circular cylinder to extrapolate that to the situation of flow past something which is of more complex shape and perhaps the shape which is more interested for aerodynamic calculations. For that, we use something which is called as conformal mapping. Now we will discuss about a little bit of introduction of conformal mapping and an example of that.

Conformal mapping in principle is nothing but a mapping of an analytic function from one complex plane to another complex plane. We will now provide an example of a conformal mapping. The objective is pretty clear that from a circular shape, we want to map to a shape which is perhaps deviating from circular but more practically relevant. There is a complex plane ζ; from there we transform it to a complex plane z as $z = \zeta + \dfrac{b^2}{\zeta}$. Let us consider that in ζ plane, we have a circle of radius b. In that case ζ will be equal to $be^{i\theta}$, i.e. $\zeta = be^{i\theta}$. Using $\zeta = be^{i\theta}$ we get $z = be^{i\theta} + be^{-i\theta}$. Now, we know that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$; we use these expressions to get $z = b(\cos\theta + i\sin\theta) + b(\cos\theta - i\sin\theta) = 2b\cos\theta$. Now if we take the x axis, then the corresponding z will be equal to $2b$ which is basically only the real component since there is no imaginary component. When θ is equal to π, then the corresponding z will be equal to $-2b$. In this way the transformation transforms the circle to a slender piece. Here lies the beauty of the conformal mapping. So, if we now want to understand the flow past a slender piece, then we can model that as a flow past a circular cylinder of radius b in the ζ plane and then apply the transformation where ζ is transformed into z.

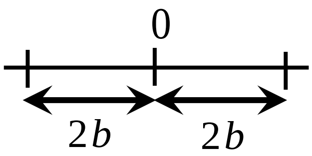

Figure 2. Transformation of a circle to a slender piece where the origin is located at (0,0). The two extremes are at a distance $2b$ on both sides of the origin which correspond to $\theta = 0$ and $\theta = \pi$ respectively.

We will provide another example before ending the discussion of this chapter. In this example we consider the radius of the circle (a) to be greater than b. In the previous example, the radius was equal to b. If the radius a is greater than b, then $\zeta = a e^{i\theta}$. Using the expression $\zeta = a e^{i\theta}$, we get the expression of z as $z = a e^{i\theta} + \frac{b^2}{a} e^{-i\theta} = a(\cos\theta + i\sin\theta) + \frac{b^2}{a}(\cos\theta - i\sin\theta) = \left(a + \frac{b^2}{a}\right)\cos\theta + i\left(a - \frac{b^2}{a}\right)\sin\theta$. Comparing this expression of z with $z = x + i\,y$ we get $x = \left(a + \frac{b^2}{a}\right)\cos\theta$ and $y = \left(a - \frac{b^2}{a}\right)\sin\theta$. We can eliminate θ by noting that $\cos^2\theta + \sin^2\theta = 1$. Then we get the equation $\frac{x^2}{\left(a + \frac{b^2}{a}\right)^2} + \frac{y^2}{\left(a - \frac{b^2}{a}\right)^2} = 1$ which represents an ellipse.

So, by this conformal mapping, we can first calculate the flow past a circular cylinder and then we can apply this conformal mapping to predict the flow behavior past an elliptic cylinder. In this way, $z = \zeta + \frac{b^2}{\zeta}$ is such a beautiful transformation that using this kind of transformation by choosing different values of the radius of the circle, we can actually transform the flow past a circular cylinder to a flow past an object which apparently is very much different in shape as compared to a circular cylinder. This is the reason why the flow past a circular cylinder is so fundamental that once we have a clear idea of flow past a circular cylinder, we can apply a transformation and analyze problems of more complex geometries. So, overall, we have discussed about various aspects of potential flow. Starting from simple elementary flows we have constructed flow past a circular cylinder. With rotation we have seen the method to calculate the lift force and the drag force and then finally we have seen the procedure to generate bodies of more complex shapes from the consideration of circular cylinder or a circle in one particular plane by using conformal mapping. In the next chapter we will discuss on a different topic.