Advanced Concepts In Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 32

Potential Flow (Contd.)

In the previous chapter we have discussed about a simulated potential flow past a circular cylinder by considering a uniform flow and a doublet superimposed together. We continue with that in the present chapter. We try to get a sketch of the streamlines around the body after which we will calculate the pressure distribution around the body. To calculate the pressure distribution on the surface of the cylinder is very critical. The reason is that as per the theory of boundary layer, the pressure distribution outside the boundary layer is imposed on the fluid on the boundary layer itself. So if we calculate the pressure distribution from potential flow consideration; that pressure distribution can be considered to be valid on a fluid where boundary layer assumptions are justified even for a case when the viscous effects are present. Although the calculation of the pressure distribution is based on potential flow consideration, this will remain valid. This is one of the big applications of the potential flow. Of course, we can use the pressure distribution up to an extent till we have the boundary layer separation. If we have the boundary layer separation then we cannot use it. So, first of all, we assume that it is a potential flow. If the density of the fluid is constant and there is no great difference in the elevation then we have $p+\frac{1}{2}\rho v^2=\text{constant}$. The assumptions behind this equation $p+\frac{1}{2}\rho v^2=\text{constant}$ are steady flow, constant density flow and irrotational flow. There is also the negligible change in the potential energy.

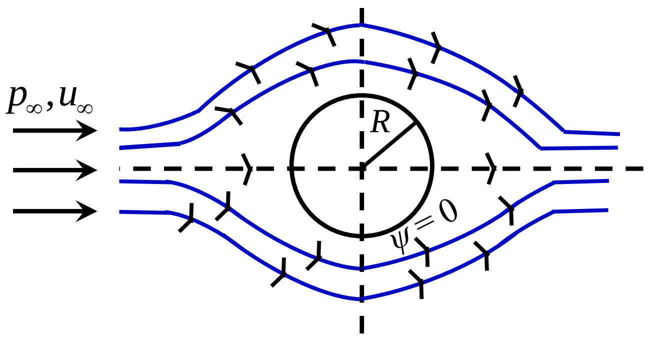

Figure 1. Streamlines for flow past a circular cylinder of radius *R*.

The pressure at the far stream is given by p_∞ and the velocity at the far stream is given by u_∞, so, $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho v^2$. The resultant v^2 is equal to $v_r{}^2 + v_\theta{}^2$ and we get $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho\left(v_r{}^2 + v_\theta{}^2\right)$. This needs to be evaluated on the surface of the cylinder (i.e. at $r = R$), so, $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho\left(v_r{}^2 + v_\theta{}^2\right)_{r=R}$. In the present example (corresponding to figure 1), at the surface of the cylinder $v_r = 0$ and the expression of v_θ is given by $v_\theta = -2\, u_\infty \sin\theta$. So the right hand side of the equation $p_\infty + \frac{1}{2}\rho u_\infty{}^2 = p + \frac{1}{2}\rho\left(v_r{}^2 + v_\theta{}^2\right)_{r=R}$ becomes equal to $p + \frac{1}{2}\rho\left(4u_\infty{}^2 \sin^2\theta\right)$. In fluid mechanics, a non-dimensional pressure is defined as the coefficient of pressure C_p which is given by $C_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho u_\infty{}^2}$. In the present example, C_p becomes $C_p = 1 - 4\sin^2\theta$.

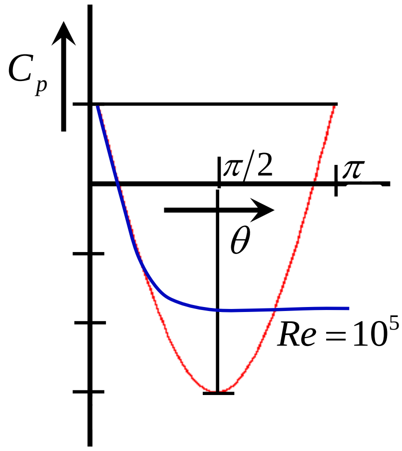

Figure 2. Variation of C_p as a function of θ as shown by the sinusoidal curve (red-colored solid line). Blue-colored solid line represents the practical scenario evaluated at $Re = 10^5$.

Now we plot the variation of C_p as a function of θ in figure 2. At $\theta = 0$, C_p is equal to 1; at $\theta = \pi/2$, C_p is equal to - 3 and at $\theta = \pi$, C_p is equal to 1. Through the three points (i.e. $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$), we draw a sinusoidal curve (shown by the red-colored solid line) as depicted in figure 2. Now question arises about the practical scenario. Let us think of the corresponding practical scenario. If the value of the Reynolds number associated with the flow is 10^5, then the variation of C_p with θ closely follows the sinusoidal variation up to $\theta = \pi/2$, then it approaches to a constant value at higher θ.

The curve corresponding to $Re = 10^5$ is shown by the blue-colored solid line in the figure. The reason of the variation of C_p approaching a constant value beyond $\theta = \pi/2$ is that we have a boundary layer separation occurring at $\theta = \pi/2$. After this point, there is a wake where we have almost a constant pressure distribution. So, up to the point where the boundary layers are separated, we observe that the pressure distribution can be successfully reproduced by the potential flow solutions. So we should not think that the potential theory is absolutely hypothetical and it has no relationship with the physical reality.

So we have seen the procedure to model a flow past a circular cylinder. Now we focus on the calculation of the drag force on the cylinder which is a matter of great engineering importance. Because of the symmetry of the flow, we can see that the drag force will be equal to zero. Whatever is the pressure distribution from the left hand side of the body (corresponding to figure 1), it will be the same from the right hand side of the body. So the horizontal components will cancel each other and that will make the drag force equal to zero. But in reality, if we subject a cylinder to a flow; there will be a drag force. Therefore, it is very important to pinpoint about that particular drag force. To get into that, we have to introduce something or some feature in addition to the uniform flow and the doublet to introduce that drag force on a cylinder. That particular thing is nothing but the viscous effect. In this analysis the viscous effect was not considered; it was considered to be a paradox and that was called as D'Alembert's paradox. It tells that although there is a fluid and a body, the net drag force is equal to zero. In reality, these theoretical calculations gave zero drag force not because of any paradox but because of the fact that the viscous effects were not considered. When the viscous effects are considered, we can see that the pressure distribution on the left hand side is different from the pressure distribution on the right hand side and that will give rise to an asymmetry (this asymmetry is also evident in the variation of C_p with θ). Next, the question arises about the lift force and the drag force.

For this lift force and the drag force, we recapitulate the earlier discussions. If we have a body of any arbitrary shape, if we have the resultant force; then we can decompose the resultant force into two parts. One part will be along the direction of flow and another part will be perpendicular to the direction of flow. In the plane we can resolve it into these two components. If it is a three-dimensional flow, we can also have a lateral thrust component. All these components are very important because in the aerodynamic situations (which tells us that how the aircrafts fly), things depend on significant control over the lift force and the drag force and also the lateral side force. The two components of the resultant force are F_D and F_L where F_D is the horizontal component or the drag force and F_L is the transverse component or the lift force respectively. Without the viscous effect, we cannot generate a drag force in this case because the symmetry in the pressure distribution cannot be broken. So the next golden question appears that without the viscous force whether we can generate the lift force or not. To examine that, one can refer to a very common practical experiment. For this, let us consider a tennis ball or a cricket ball (i.e. a body of spherical shape). So, in the fluid where the viscous effects are even negligible, if we approximate the cylinder or the sphere (corresponding to the shape of the ball); if we spin it clockwise then we find that there is a tendency of it to go up or down depending on the direction of this spin. This is called as top spin in a sports ball and this is due to an effect called as Magnus effect. So question arises about this effect which can give rise to a lift force where everything is absolutely symmetric. Intuitively we do not expect a lift force. In fact, in the present example, we do not get a lift force. Just like the drag force where the two horizontal components from the left hand side and the right hand side cancel each other, in case of lift force, the two vertical components from the top and the bottom cancel each other. But if we bring in a rotation, rotation can break the symmetry not for the drag force but for the lift force. So, rotation in the paradigm of a potential flow is a type of irrotational or free vortex (as discussed in the kinematics chapter). Now we will consider a rotation and to bring in a rotation, the natural choice is a particular type of flow which is a free vortex flow. We cannot bring in a forced vortex flow in the picture here because that is not an irrotational flow and we are talking about the irrotational flow here. So, next we will study our next example which is vortex. Although we say it vortex, this is actually irrotational vortex; we have to specify this very clearly that it is irrotational or free vortex.

Free vortex is defined as $v_\theta = \dfrac{c}{r}$ and $v_r = 0$. So the question appears about the corresponding complex potential function. That is the question because we have to superimpose this with the uniform flow and doublet to give a rotation to the circular cylinder. We have $\dfrac{dF}{dz} = \left(v_r - i\,v_\theta\right) e^{-i\theta}$; for a free vortex flow it becomes

$\frac{dF}{dz} = -\frac{i\,c}{r\,e^{i\theta}} = -\frac{i\,c}{z}$. Now if we integrate it with respect to *z*, we get $F = -i\,c\,\ln z$ where c is an arbitrary constant. We should relate c with the strength of the rotation which is also called the circulation. The circulation (Γ) is defined as the contour integral of $\vec{v}\cdot d\vec{l}$ over a closed path, i.e. $\Gamma = \oint \vec{v}\cdot d\vec{l}$. In the present example, it becomes $\Gamma = \int_{\theta=0}^{\theta=2\pi} v_\theta\, r\, d\theta$. Here $v_\theta = c/r$ and $dl = r\,d\theta$, so r gets cancelled from the numerator and the denominator; we get rid of the singularity and we finally get $\Gamma = 2\pi c$ or, $c = \frac{\Gamma}{2\pi}$. Substituting $c = \frac{\Gamma}{2\pi}$ in the expression of $\frac{dF}{dz}$, we get $\frac{dF}{dz} = -\frac{i\Gamma}{2\pi z}$ and similarly, $F = -\frac{i\Gamma}{2\pi}\ln z$. When Γ is positive it indicates the anti-clockwise rotation. The minus sign is already there in the expression of *F*, if it is negative it shows anti-clockwise rotation. If it is positive it shows clockwise rotation. We have to keep this in mind because we have to physically interpret some of the results.

Now we will come to the question that what will be result of the superposition of uniform flow, doublet and free vortex. The combination of uniform flow and doublet is physically representative of a flow past a circular cylinder when the cylinder is not rotating. The consideration of free vortex flow adds a rotation to the cylinder. So this physically mimics the potential flow past a rotating circular cylinder. In the present case, the complex potential function is the combination of uniform flow along *x* axis $u_\infty z$, doublet $\frac{m}{z}$ and free vortex $-\frac{i\Gamma}{2\pi}\ln z$, so, $F = u_\infty z + \frac{m}{z} - \frac{i\Gamma}{2\pi}\ln z$. We have already seen that $m = u_\infty R^2$ to represent the flow past a circular cylinder of radius *R*, so, $F = u_\infty z + \frac{u_\infty R^2}{z} - \frac{i\Gamma}{2\pi}\ln z$. Now with introduction of uniform flow and doublet, we have ensured that there is no penetration boundary condition at the surface of the cylinder. But with the introduction of free vortex flow, this is no more assured.

We can assure this by adding a constant $a + i\,b$ to the expression of the complex potential function *F* such that it becomes $F = u_\infty z + \frac{u_\infty R^2}{z} - \frac{i\Gamma}{2\pi}\ln z + a + i\,b$. In that case we can choose '*a'* and *'b*' as per our freedom. This $a+ib$ is just a constant and therefore it will not reflect in terms of velocity. It will just alter the values of the stream function and the velocity potential but not their derivatives. Since $a+ib$ is a constant and velocities are derivatives of stream function and velocity potential, it will not alter the velocity. But it will allow us to adjust the velocity in such a way that the radial component of velocity is zero on the surface of the body.

Now substituting $z=re^{i\theta}$ in the expression of the complex potential function F we get $F=u_\infty re^{i\theta}+\frac{u_\infty R^2}{r}e^{-i\theta}-\frac{i\Gamma}{2\pi}\ln r+\frac{\Gamma}{2\pi}\theta+a+ib$ where $\ln z=\ln\left(re^{i\theta}\right)=\ln r+i\theta$ and $i^2=-1$ have been used. We can now separate the real part and the imaginary part. The real part of F is given by $\left(u_\infty r+\frac{u_\infty R^2}{r}\right)\cos\theta+\frac{\Gamma}{2\pi}\theta+a$ and the imaginary part is given by $u_\infty r\sin\theta-\frac{u_\infty R^2}{r}\sin\theta-\frac{\Gamma}{2\pi}\ln r+b$. So, the complex potential F can be represented as $F=\left[\left(u_\infty r+\frac{u_\infty R^2}{r}\right)\cos\theta+\frac{\Gamma}{2\pi}\theta+a\right]+i\left[u_\infty r\sin\theta-\frac{u_\infty R^2}{r}\sin\theta-\frac{\Gamma}{2\pi}\ln r+b\right]$. Again from definition, we know that $F=\phi+i\psi$, so, $\phi=\left(u_\infty r+\frac{u_\infty R^2}{r}\right)\cos\theta+\frac{\Gamma}{2\pi}\theta+a$ and $\psi=u_\infty r\sin\theta-\frac{u_\infty R^2}{r}\sin\theta-\frac{\Gamma}{2\pi}\ln r+b$. So, to set up a no-penetration boundary condition with a reference streamline on the surface of the cylinder we must have $\psi=0$ at $r=R$. $\psi=0$ at $r=R$ means that $\psi=\left(u_\infty R-u_\infty R\right)\sin\theta-\frac{\Gamma}{2\pi}\ln R+b=0$. The first part $\left(u_\infty R-u_\infty R\right)\sin\theta$ was already zero for the consideration of uniform flow and doublet. So we can set b as $b=\frac{\Gamma}{2\pi}\ln R$. In this way we can model the consideration of uniform flow, doublet and free vortex with the particular choice of '*b*' with an arbitrary choice of '*a*'. For '*a*', we can make a choice arbitrarily because although we have a constraint on the steam function we do not have a constraint on the velocity potential. Constraint on the stream function is also up to us. We may choose of course a non-zero stream function on the surface of the body. There is nothing wrong with it but customarily we choose it to be equal to zero otherwise that brings in an extra constraint.

So the potential flow past a rotating circular cylinder can be represented by the generic form $F = u_\infty z + \frac{u_\infty R^2}{z} - \frac{i\Gamma}{2\pi} \ln z + a + i b$. Of course, the matter of primary interest is to see the velocity distribution. But a matter of more intense primary interest is to see the lift force and the drag force. The reason is that the lift force and the drag force are the things which are matter of concern in engineering. We can calculate these forces using a very powerful theorem called as Blasius force theorem. It can even take a body of any arbitrary shape as long as the body has a closed contour. In the next chapter we will discuss the steps to find out the lift force and the drag force on any arbitrary shaped body for a given complex potential using the Blasius force theorem.