Advanced Concepts In Fluid Mechanics

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Lecture – 31
Potential Flow (Contd.)

In the previous chapter we have discussed about the doublet and we will continue this in the present chapter. If we recall from the previous chapter, the complex potential function F for the doublet is given by $F=\dfrac{m}{z}$. From definition, the complex potential function F is given by $F=\phi+i\psi$ and the derivative of F is written as $\dfrac{dF}{dz}=u-iv$. In terms of r-θ coordinate, the derivative $\dfrac{dF}{dz}$ can be written as $\dfrac{dF}{dz}=\left(v_r-iv_\theta\right)e^{-i\theta}$ (we have derived this expression in the previous chapter). So, the problem in terms of complex analysis is to separate this expression into real and imaginary parts. Real part represents the velocity potential and imaginary part represents the stream function. Using $z=x+iy$ the expression $F=\dfrac{m}{z}$ is rewritten as $F=\dfrac{m}{x+iy}$. Now we need to make the denominator real; to do this we have to multiply both the numerator and the denominator by $x-iy$. Then we get $F=\dfrac{m\left(x-iy\right)}{\left(x+iy\right)\left(x-iy\right)}$ where the denominator becomes equal to x^2+y^2 because $i^2=-1$, so, $F=\dfrac{m\left(x-iy\right)}{x^2+y^2}$. Comparing the expressions of $F=\phi+i\psi$ and $F=\dfrac{m\left(x-iy\right)}{x^2+y^2}$ we get $\phi=\dfrac{mx}{x^2+y^2}$ and $\psi=\dfrac{-my}{x^2+y^2}$. Now let us try to draw $\phi=\text{constant}$ lines and $\psi=\text{constant}$ lines. To draw $\phi=\text{constant}$ lines we need to obtain the corresponding equation. The equation corresponding to $\phi=\text{constant}$ is given by $x^2+y^2-\dfrac{mx}{\phi}=0$. This equation can be rewritten as $x^2-\dfrac{2m}{2\phi}x+\left(\dfrac{m}{2\phi}\right)^2+y^2=\left(\dfrac{m}{2\phi}\right)^2$ or, $\left(x-\dfrac{m}{2\phi}\right)^2+y^2=\left(\dfrac{m}{2\phi}\right)^2$. This equation is as good as the form $\left(x-a\right)^2+y^2=r^2$ which represents the equation of a circle at centre (a,0) and radius r. So the equation

$\left(x-\frac{m}{2\phi}\right)^2+y^2=\left(\frac{m}{2\phi}\right)^2$ is the equation of a circle with centre $\left(\frac{m}{2\phi},0\right)$ and radius $\frac{m}{2\phi}$.

We will now try to make a sketch of this equation.

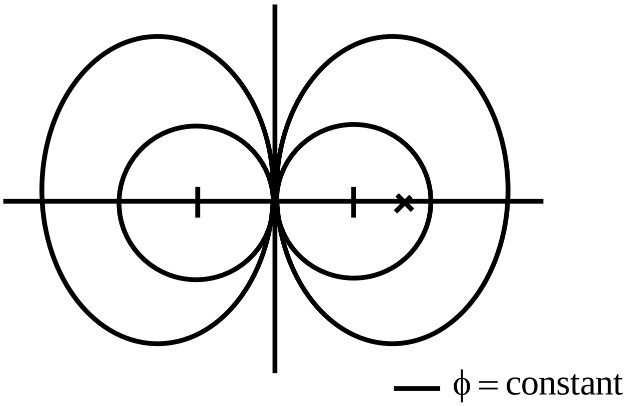

Figure 1. The pictorial depiction of $\phi=\text{constant}$ lines which represent the equation of circle with centre $\left(\frac{m}{2\phi},0\right)$ and radius $\frac{m}{2\phi}$.

The value of ϕ can be both positive and negative. Let us take one particular point for which the *x*-coordinate is $\frac{m}{2\phi}$ and the *y*-coordinate is 0. Also the radius of the circle is equal to $\frac{m}{2\phi}$. The corresponding curve is shown by the small circle on the right hand side of figure 1. Now if the center of the circle is shifted on the right hand side, it will also result a bigger circle. These circles represent the scenario for positive value of ϕ. For negative values of ϕ, the circles will be on the left side of the figure. These circles will be the mirror images with respect to the *y* axis. All these circles (black-colored) represent $\phi=\text{constant}$ lines. In order to ease the understanding, we will now draw $\psi=\text{constant}$ lines with a different color.

To draw $\psi=\text{constant}$ lines, the corresponding equation for $\psi=\text{constant}$ is given by $x^2+y^2+\frac{my}{\psi}=0$. This equation can be rewritten as $x^2+y^2+\frac{2m}{2\psi}y+\left(\frac{m}{2\psi}\right)^2=\left(\frac{m}{2\psi}\right)^2$

or, $x^2+\left(y+\frac{m}{2\psi}\right)^2=\left(\frac{m}{2\psi}\right)^2$. This represents the equation of a circle with centre

$\left(0,-\frac{m}{2\psi}\right)$ and radius $\frac{m}{2\psi}$.

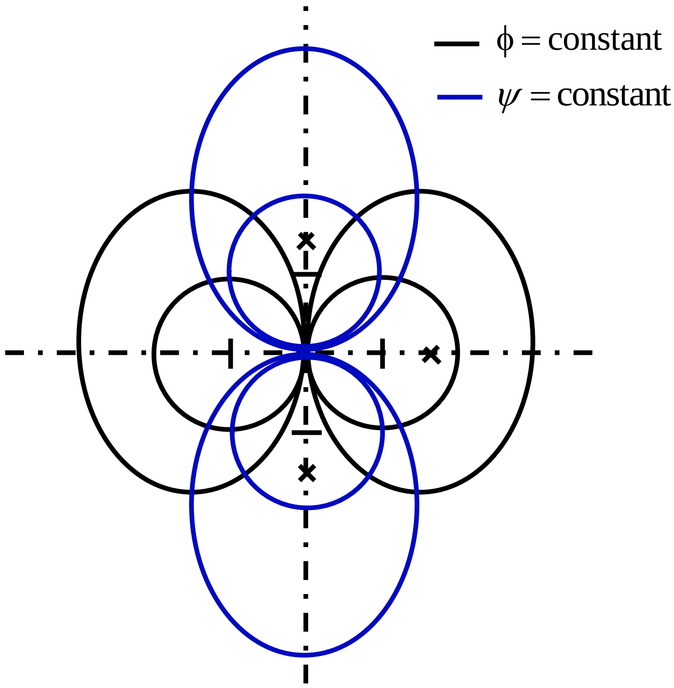

Figure 2. The pictorial depiction of both $\phi = \text{constant}$ lines and $\psi = \text{constant}$ lines. $\phi = \text{constant}$ lines have been already discussed in figure 1. $\psi = \text{constant}$ lines represent the equation of circle with centre $\left(0,-\frac{m}{2\psi}\right)$ and radius $\frac{m}{2\psi}$.

The corresponding circles for $\psi = \text{constant}$ lines are shown by the blue-colored circles in figure 2. From this figure we can see that if we know one of the two equations corresponding to $\phi = \text{constant}$ and $\psi = \text{constant}$, we can clearly draw the other from the consideration that they are orthogonal to each other. So these circles wherever intersect, if we draw common tangent, they must be orthogonal.

So far we have discussed about uniform flow, Rankine oval, source and sink individually and combination of source and sink (which we call as doublet). Up to this step we have done exercise which is primarily of academic nature. One may argue that since the source and the sink are located at a certain distance apart, what kind of practical flow can be represented by this example. If it does not represent the practical flow it is not of that interest. Let us now imagine that instead of this kind of flow, we want to model a practical flow over a body of the shape of a cylinder. Cylinder, of course, is not the most general shape but cylinder is ideal or an idealized geometrical shape and streamlines and equipotential lines for flow past a cylinder can give a qualitative understanding of flow of a fluid over bodies of much more complicated shapes. One of the elementary things for studying aerodynamics is to first understand the flow past a circular cylinder. So we

will start with our objective of generating the shape of a body through the superposition of complex potentials which represent the flow past a circular cylinder. We will now show that the nice combination of uniform flow and doublet can achieve this feat. We will discuss the procedure to show that the consideration of uniform flow and doublet can be used to model the potential flow past a circular cylinder. To do that, we will focus on the complex potential function *F*. There is a great advantage because of the linearity of the problem. If $F = F_1$ is the complex potential for the uniform flow and $F = F_2$ is the complex potential for the doublet then for the combination, the complex potential is $F = F_1 + F_2$. F is equal to $\phi + i\psi$ where both ϕ and ψ satisfy the Laplace equation which is a linear second order partial differential equation for this two dimensional irrotational flow. Uniform flow means that it could be uniform along any direction; let us assume that the uniform flow is along the *x* direction which is given by $u_\infty z$ while the expression for doublet is given by $\frac{m}{z}$ where m is the strength of the doublet, so, $F = u_\infty z + \frac{m}{z}$. We have to actually choose this parameter m; at this stage we are keeping it open. Using $F = u_\infty z + \frac{m}{z}$ we get $\frac{dF}{dz} = u_\infty - \frac{m}{z^2}$. Again, $\frac{dF}{dz}$ is also defined as $\frac{dF}{dz} = (v_r - i v_\theta) e^{-i\theta}$. If we express $u_\infty - \frac{m}{z^2}$ in this particular form of $(v_r - i v_\theta) e^{-i\theta}$, then we get $\frac{dF}{dz} = \left[u_\infty e^{i\theta} - \frac{m}{r^2} e^{-i\theta} \right] e^{-i\theta}$ where we have substituted $z = r e^{i\theta}$. Next we use the expansions $e^{i\theta} = \cos\theta + i \sin\theta$ and $e^{-i\theta} = \cos\theta - i \sin\theta$ because we have to somehow separate the v_r part and the v_θ part. We get $\frac{dF}{dz} = \left[u_\infty (\cos\theta + i\sin\theta) - \frac{m}{r^2}(\cos\theta - i\sin\theta) \right] e^{-i\theta} = \left[\left(u_\infty - \frac{m}{r^2} \right)\cos\theta + \left(u_\infty + \frac{m}{r^2} \right) i \sin\theta \right] e^{-i\theta}$.

If we compare this expression with $\frac{dF}{dz} = (v_r - i v_\theta) e^{-i\theta}$ then we get $v_r = \left(u_\infty - \frac{m}{r^2} \right)\cos\theta$ and $v_\theta = -\left(u_\infty + \frac{m}{r^2} \right)\sin\theta$.

Now let us imagine the flow past a circular cylinder. For flow past a circular cylinder we need to satisfy the basic boundary condition at the surface of the cylinder. The surface of the cylinder is defined by a constant radius. On the surface of the cylinder we must have no penetration boundary condition which means that at $r = R$(which is the radius of the cylinder), we must have $v_r = 0$. This is called as the no penetration boundary condition. No penetration boundary condition is very important because it does not depend on whether it is potential flow or not. It is a kind of kinematic boundary condition at the surface. So kinematically there cannot be any penetration across the surface until and unless there is a hole. Overall, this is what is being represented by the no penetration boundary condition. So, for the no penetration boundary condition we must have $u_\infty - \frac{m}{R^2} = 0$ at $r = R$ such that at $r = R$, v_r can be equal to zero. So we can set the strength of the doublet m as $m = u_\infty R^2$ which completes the first part of analysis. Then question arises about the other velocity component v_θ. At $r = R$, $-v_\theta\big|_{r=R} = \left(u_\infty + \frac{m}{R^2}\right)\sin\theta$ and $m = u_\infty R^2$, so, $-v_\theta\big|_{r=R} = 2u_\infty \sin\theta$. Interestingly, v_θ is the tangential velocity on the surface of a cylinder. By setting $v_\theta = -2u_\infty \sin\theta$ at the surface of the cylinder we are allowing slip. So we can clearly understand that where the viscous effects are not present, slip must be allowed otherwise we cannot satisfy the requirements of continuity and momentum conservation on the surface. Before discussing about v_r and v_θ, it may be interesting to draw the streamlines.

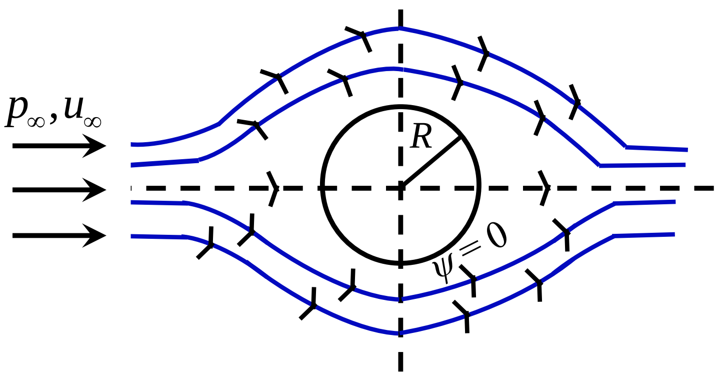

Figure 3. Streamlines for flow past a circular cylinder of radius R with the consideration of uniform flow and doublet.

The expression of F is given by $F = u_\infty z + \frac{m}{z}$ which upon substitution of $z = r e^{i\theta}$ becomes $F = u_\infty r e^{i\theta} + \frac{m}{r} e^{-i\theta}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, we get $F = u_\infty r(\cos\theta + i\sin\theta) + \frac{m}{r}(\cos\theta - i\sin\theta) = \left(u_\infty r + \frac{m}{r}\right)\cos\theta + \left(u_\infty r - \frac{m}{r}\right)\sin\theta i$. Now we look into the expressions of the velocities v_r and v_θ also. $v_r = 0$ for all locations on the cylinder because that is how it is designed. $v_\theta = -2u_\infty \sin\theta$ will be equal to zero for $\theta = 0$ and $\theta = \pi$. So, at $\theta = 0$ and $\theta = \pi$, both the velocities v_r and v_θ are equal to zero which means that the resultant velocity will be equal to zero. These points are called as stagnation points. Substituting $m = u_\infty R^2$, we get $F = \left(u_\infty r + \frac{u_\infty R^2}{r}\right)\cos\theta + \left(u_\infty r - \frac{u_\infty R^2}{r}\right)\sin\theta i = \phi + i\psi$. First of all, when r is equal to R, it represents the surface of the cylinder. The cylinder is shown by the black-colored circular portion in figure 3. The uniform flow at the far stream is represented by u_∞. From the expression $\psi = \left(u_\infty r - \frac{u_\infty R^2}{r}\right)\sin\theta$ we can understand that when $r = R$ (i.e. at the surface), ψ is equal to zero. Therefore, the surface of the streamline is itself a streamline with $\psi = 0$. Depending on whether r is greater than R or smaller than R, we will get two different types of streamlines. In one case the term $\frac{u_\infty R^2}{r}$ will be greater than $u_\infty r$ while in the other case $u_\infty r$ will be greater than $\frac{u_\infty R^2}{r}$. So, $r < R$ and $r > R$ will be two different cases. We have to now decide that out of these two cases which case will be interesting. The case with $r > R$ is interesting because we are interested to model the flow external to a circular cylinder. What happens inside the cylinder is not of our interest. So, flow inside the circular cylinder is like artificial, it is not of a matter of interest. Of course, there will also exist v_r and v_θ components (or, ϕ, ψ etc.) but in reality, we usually have a blocked cylinder and flow inside the cylinder is not of great importance. We recall the expression of ψ which reads as $\psi = \left(u_\infty r - \frac{u_\infty R^2}{r}\right)\sin\theta$. We can clearly see that ψ will be equal to zero at $\theta = 0$ and $\theta = \pi$. At $\theta = 0$ and $\theta = \pi$, the streamlines should be straight lines (represented by the dotted straight line in figure 3 which is passing through the centre of the cylinder). Streamlines should be representing $\psi = 0$ irrespective of the value of *r* as long as θ is equal to 0 or π. If θ is not equal to 0 or π, then the representative streamlines are shown by the blue-colored lines in figure 3. From these lines we see that the streamlines look very symmetric.

This symmetry can be broken if the viscous effects are brought in; then this symmetry breaking phenomenon due to viscous effects can give rise to a drag force on the cylinder. However, without considering the viscous effect we can get significant insights on the pressure distribution on the cylinder at least up to the region over which these kind of streamline pattern is maintained. If there is viscous effect, then there can be a phenomenon called as flow separation at the back face of the cylinder which is also called as wake.

In that scenario, this nature of the streamline patterns on the back side may be disturbed. But in the front side, this nature of the streamlines will be maintained to a large extent at least up to $\theta = 90^0$. Then there is likely to be a remarkable agreement between the pressure distribution coming out of the potential flow solution and the pressure distribution coming out of the viscous flow solution (which represents a more realistic scenario). This agreement will be valid up to which this kind of streamline pattern is maintained. Therefore, getting an expression of the pressure distribution around the body is a matter of great importance for us which will be discussed in the next chapter.