Advanced Concepts In Fluid Mechanics
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Lecture - 30
Potential Flow (Contd.)

In the previous chapter we discussed about two elementary types of flows; one is the uniform flow while the other one is a source or a sink. Let us try to make an analysis of whether we can superimpose these types of flows. Let us take an example; it is the third example for us which is an example of uniform flow along the *x* axis as well as the presence of a source at $x = 0$.

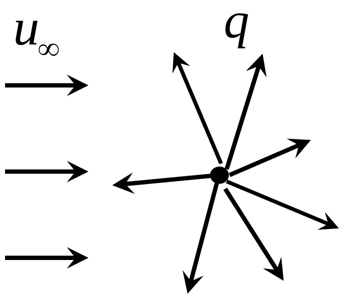

Figure 1. Physical picture of the example where there is uniform flow u_∞ as well as strength q of a source located at the origin.

The physical picture of the problem is depicted in figure 1 where we have uniform flow u_∞ as well as q as the strength of the source located at the origin. Now question arises whether by looking into the flow we can guess the possibility of the presence of a stagnation point somewhere. Stagnation point is the point where the velocity is equal to zero. The uniform flow (denoted by u_∞) is going towards the positive *x* direction and some of the flow (on the left hand side of flow around the point source) is going towards the negative *x* direction. So these two opposing flows should cancel each other somewhere and therefore, we have a chance of getting a stagnation point along the *x* axis. So the first observation is that we have a chance of getting a stagnation point along the *x* axis or on the *x* axis.

In this scenario, the complex potential function F is given by $F = u_\infty z + \frac{q}{2\pi}\ln z$ and therefore, $\frac{dF}{dz} = u_\infty + \frac{q}{2\pi z}$. According to the definition of the complex potential function we can write $\frac{dF}{dz} = u - iv$. We can also express it in terms of $r-\theta$ coordinates sometimes. Since there is a source in the present example, it is easier to express it in

terms of $r-\theta$ coordinate. Then we can rewrite $\frac{dF}{dz}$ as $\frac{dF}{dz} = u_\infty + \frac{q}{2\pi r}e^{-i\theta} = \left(u_\infty e^{i\theta} + \frac{q}{2\pi r}\right)e^{-i\theta}$. The reason of expressing $\frac{dF}{dz}$ in this way is that we can then get the expressions of the velocities v_r and v_θ. The expressions of v_r and v_θ are given by $v_r = u_\infty \cos\theta + \frac{q}{2\pi r}$ and $v_\theta = -u_\infty \sin\theta$ respectively. At the stagnation point both v_r and v_θ are equal to zero. Our objective is to find out the stagnation point. The streamline passing through the stagnation point is called as the stagnation streamline. The streamline passing through the stagnation point will represent the shape of the body first passed which $\frac{dF}{dz}$ indicates the resultant flow. At the stagnation point we have $v_\theta = 0$. We need to find the possible location of the stagnation point. It cannot be located on the right hand side of the origin but it can be located on the left hand side of the origin where the negative flow and the positive flow can cancel each other. That left hand side or the negative side of the x axis is represented by $\theta = \pi$. Substituting $\theta = \pi$ at $v_r = 0$ we get $u_\infty \cos\pi + \frac{q}{2\pi r} = -u_\infty + \frac{q}{2\pi r} = 0$, so, $r = \frac{q}{2\pi u_\infty}$.

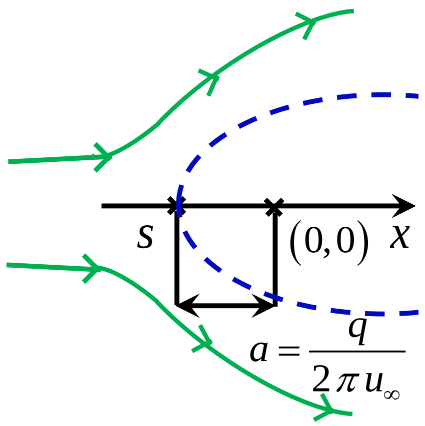

Figure 2. Pictorial depiction of the present example with source at the origin and point '*s*' being the stagnation point. The streamline passing through the stagnation point is shown by the blue-colored dotted line. The streamline outside the body is shown by the green-colored solid line.

Now we pictorially represent this in figure 2 where the source is located at the origin.

The radial distance $r = \dfrac{q}{2\pi u_\infty}$ on the left hand side of the origin is located in the figure. The corresponding value of the angle $\theta = \pi$ is also located in the figure. The stagnation point is marked by the point '*s*' in the figure. Once we have the stagnation point our objective will be to obtain the stagnation streamline. We have $F = u_\infty z + \dfrac{q}{2\pi}\ln z$; we are trying to extract ψ from here (somehow). The expression $F = u_\infty z + \dfrac{q}{2\pi}\ln z$ is nothing but $F = u_\infty r e^{i\theta} + \dfrac{q}{2\pi}\ln\left(r e^{i\theta}\right)$; $e^{i\theta}$ is equal to $\cos\theta + i\sin\theta$ and $\ln\left(r e^{i\theta}\right)$ is equal to $\ln r + i\theta$. So we get $F = u_\infty r\left(\cos\theta + i\sin\theta\right) + \dfrac{q}{2\pi}\ln r + i\dfrac{q}{2\pi}\theta$ which is also equal to $\phi + i\psi$, i.e. $F = u_\infty r\left(\cos\theta + i\sin\theta\right) + \dfrac{q}{2\pi}\ln r + i\dfrac{q}{2\pi}\theta = \phi + i\psi$.

Now if we isolate the real and the imaginary parts, then $\psi = u_\infty r\sin\theta + \dfrac{q}{2\pi}\theta$. The next question arises about the shape of the body. The representative flow is the uniform flow in addition with the point source at the origin. The shape of the body will be governed by the streamline. The streamline will pass through the stagnation point because the stagnation point is also located on the body. One may argue that if the stagnation point is on the left side of the origin then whether there is any other stagnation point on the body. Also question arises about whether all the points on the body can be considered as the stagnation points.

Here lies a very interesting difference between the real viscous flow and the potential flow. In the real viscous flow all the points on the body will have no slip and no penetration boundary conditions. The no slip boundary condition comes from the consideration where viscosity plays a very critical role. The no penetration boundary condition is a kinematic boundary condition which comes from the body contour; it tells that the fluid cannot penetrate through the body. For the potential flow the no penetration boundary condition is present. So no penetration boundary condition is much more fundamental than the no slip boundary condition. It is present irrespective of whether it is irrotational flow or rotational flow. But the no slip boundary condition may be violated which means that all the points on the body will not have zero velocity. Although no penetration condition will be valid but there may be slip condition except the stagnation point or the stagnation points. There can be multiple stagnation points where both the velocity components will be equal to zero. This is the reason why the stagnation point is so special in a potential flow. Now the shape of the body (whatever be the shape) should pass through the stagnation point because the stagnation point is one of the points where the flow is towards standstill. We define the stagnation streamline as $\psi = \psi_s$. Let $r = a = \frac{q}{2\pi u_\infty}$, then, $\psi_s = u_\infty a \sin\theta + \frac{q}{2\pi}\theta$. Substituting $\theta = \pi$ we get $\psi_s = u_\infty a \sin\pi + \frac{q}{2\pi}\pi = \frac{q}{2}$. This is the equation of the streamline passing through the stagnation point. Mathematically we can write this equation but it is always interesting to imagine physically that what can be the shape coming out of this equation (this is very important). This is a priceless skill to imagine in that way and this is where there is a paradigm shift of time. In earlier days, students were always encouraged to sketch a given mathematical function such that they can visualize the function. But nowadays if a mathematical function is provided, students have software which can plot any mathematical function immediately and get the picture. So it is completely missing that how to speculate the shape of the curve (curve, line whatever) from a given equation.

We have our equation $\frac{q}{2} = u_\infty r \sin\theta + \frac{q}{2\pi}\theta = u_\infty y + \frac{q}{2\pi}\theta$ where we have substituted $y = r\sin\theta$, so, $u_\infty y = \frac{q}{2} - \frac{q}{2\pi}\theta = \frac{q}{2}\left(1 - \frac{\theta}{\pi}\right)$ or $y = \frac{q}{2\,u_\infty}\left(\frac{\pi-\theta}{\pi}\right)$. Here we can substitute different values of θ. We now focus on drawing the curve of this equation. First of all, the curve should pass through the stagnation point '*s*'. Now question arises about the scenario at $x \to \infty$. We should have $y = 0$ somewhere on the *x* axis which basically occurs at the stagnation point corresponding to the value of $\theta = \pi$. Apart from this point the curve does not pass through the *x* axis again (except point '*s*'). Additionally the curve should be symmetric with respect to the *x* axis. Now we can change the value of θ from $\theta = \pi$ to $\theta = \pi/2$. We can again substitute $r\sin\theta$ as y to get $r\sin\theta = \frac{q}{2\,u_\infty}\left(\frac{\pi-\theta}{\pi}\right)$ such that we can understand it better. So let us take $\theta = \pi/2$ as an example. When $\theta = \pi/2$, we have $r = \frac{q}{2\,u_\infty}\left(\frac{\pi-\pi/2}{\pi}\right)$. In this way we can get points with different *r* and different θ and the curve will be symmetric about the *x* axis. To check whether the curve is symmetric about x axis or not we need to check the expression of *y* which reads as $y = \frac{q}{2\,u_\infty}\left(\frac{\pi-\theta}{\pi}\right)$. If there is no presence of *x* in the expression of *y* then we can say that the curve is symmetric about the *x* axis. We can do all the checks before drawing the curve passing through the stagnation point. Sometimes writing in terms of polar coordinates makes it little bit difficult in terms of visualization. So we can write it in terms of rectangular Cartesian coordinates. Then the angle θ can be replaced by $\theta = \tan^{-1}(y/x)$. The curve passing through the stagnation point is shown by the blue-colored dotted line in figure 2. Now we need to know about what happens at $x \to \infty$. At $x \to \infty$, we will find that the slope of the curve will be equal to zero. So the asymptote will be parallel to the *x* axis for both + *y* and – *y*. The shape of the curve is a beautiful shape which is called as the half body or the Rankine half body in a potential flow theory. The streamline outside the body is shown by the green-colored solid line. There will also be streamlines inside the body but the entire exercise is done actually to mimic the flow outside the body. So in the inside we will have the source and there will be uniform flow; something will happen with the superposition of all factors. But the matter of importance is the shape of the body and at any given value of θ, we can obtain the height h by using the formula $u_\infty\, y = \frac{q}{2}\left(1-\frac{\theta}{\pi}\right)$. When θ is equal to zero, all lines on the x axis will correspond to $\theta = 0$. So when $\theta = 0$, we can see that the curve is asymptotically attaining a height which is equal to $\frac{q}{2\pi\, u_\infty}$. Also, if we put $\theta = 0$ in the expression $u_\infty\, y = \frac{q}{2}\left(1-\frac{\theta}{\pi}\right)$ we can observe that the asymptotic height $\frac{q}{2\pi\, u_\infty}$ (the height of the body) is recovered. All these geometric parameters are important because these geometric parameters can only tell us about the shape of the body past which the flow is generated by this kind of arrangement. We will consider another superposition before concluding the present chapter.

This was our third example on the potential flow; now we will consider the fourth example. We consider a source of strength q at $x = -\varepsilon$ as well as a sink of strength q at $x = \varepsilon$ where $\varepsilon \to 0$. We will now start with this example; we will not be able to finish this example within the present chapter and therefore it will be continued in the next chapter.

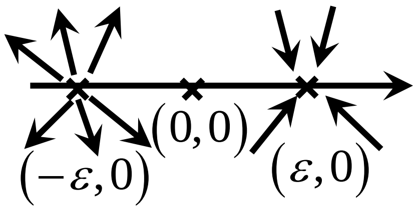

Figure 3. Physical picture of the fourth example where there is source of strength q at $x = -\varepsilon$ as well as a sink of strength q at $x = \varepsilon$.

The physical picture of the fourth example is shown in figure 3. Here, the origin is located at (0,0). The sink of strength q is located at (ε,0) while the source of strength q is located at (- ε,0). Sink means all the flow is coming towards it while source means all the flow is going away from it. For this situation, the complex potential function F will be the superposition of the individual complex functions corresponding to the source and the sink. In a generic representation F is written as $F = \frac{q}{2\pi}\ln\left(z-a\right)$ with a indicating the shift in the position. For the source, F is given by $F = \frac{q}{2\pi}\ln\left(z+\varepsilon\right)$ where the shift in the position from the origin is $-\varepsilon$. For the sink F is given by $F = \frac{-q}{2\pi}\ln\left(z-\varepsilon\right)$ where q is replaced by – q because of the flow occurring in the reverse direction. So, overall, the complex potential function F for the present example is given by $F = \frac{q}{2\pi}\ln\left(z+\varepsilon\right) - \frac{q}{2\pi}\ln\left(z-\varepsilon\right)$. We can also write this expression of F as $F = \frac{q}{2\pi}\ln\left(z\right) + \frac{q}{2\pi}\ln\left(1+\varepsilon/z\right) - \frac{q}{2\pi}\ln\left(z\right) - \frac{q}{2\pi}\ln\left(1-\varepsilon/z\right)$. So, $\frac{q}{2\pi}\ln\left(z\right)$ gets cancelled from this expression and F becomes $F = \frac{q}{2\pi}\ln\left(1+\varepsilon/z\right) - \frac{q}{2\pi}\ln\left(1-\varepsilon/z\right)$. We can expand $\ln\left(1+\varepsilon/z\right)$ in a logarithmic series if ε is small. Using the expansion of $\ln\left(1+x\right) = x - x^2/2 + x^3/3 + ..$ we get $F = \frac{q}{2\pi}\left[\left(\frac{\varepsilon}{z}\right) - \left(\frac{\varepsilon}{z}\right)^2 + ..\right] - \frac{q}{2\pi}\left[\left(-\frac{\varepsilon}{z}\right) - \left(\frac{\varepsilon}{z}\right)^2 + ..\right]$. So the term $\frac{q}{2\pi}\left(\frac{\varepsilon}{z}\right)^2$ gets cancelled and we get $F = \frac{q}{2\pi}\frac{\varepsilon}{z} + \frac{q}{2\pi}\frac{\varepsilon}{z} = \frac{q}{2\pi}\frac{2\varepsilon}{z} = \frac{q\varepsilon}{\pi z}$ in the limit $\varepsilon \to 0$. Just in a short hand, if we write $m = \frac{q\varepsilon}{\pi}$ then $F = \frac{m}{z}$. This is a new potential that we have realized. If we bring the source and the sink very close to each other then this is called as doublet. Therefore question arises that what kind of body doublet generates, what kind of streamlines will be generated and what will be the potential lines. All these things will be discussed in the next chapter.

Overall we have looked into the uniform flow with source and sink with the latest addition is the consideration of doublet. We have not completed the doublet part because it requires a good inspection of the potential and the streamlines to get a picture of the flow field in a doublet. This part will be covered in the next chapter.