Advanced Concepts In Fluid Mechanics
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Lecture - 03
Angular Deformation of Fluid Elements

I. Angular Deformation of Fluid

Fluid deformation can be categorized into two parts – angular deformation, and, linear deformation. Notwithstanding the nature of deformation (angular or linear), fluids by their nature undergo continuous deformation so long a force is being applied. Therefore, the absolute magnitude of deformation does not matter for a fluid, rather the rate of deformation does, which is mathematical quantified as rate of strain.

We will first study angular deformation and study linear deformation in the next lecture. While angular deformation is generally three-dimensional, it can be decomposed into components along the three two-dimensional planes.

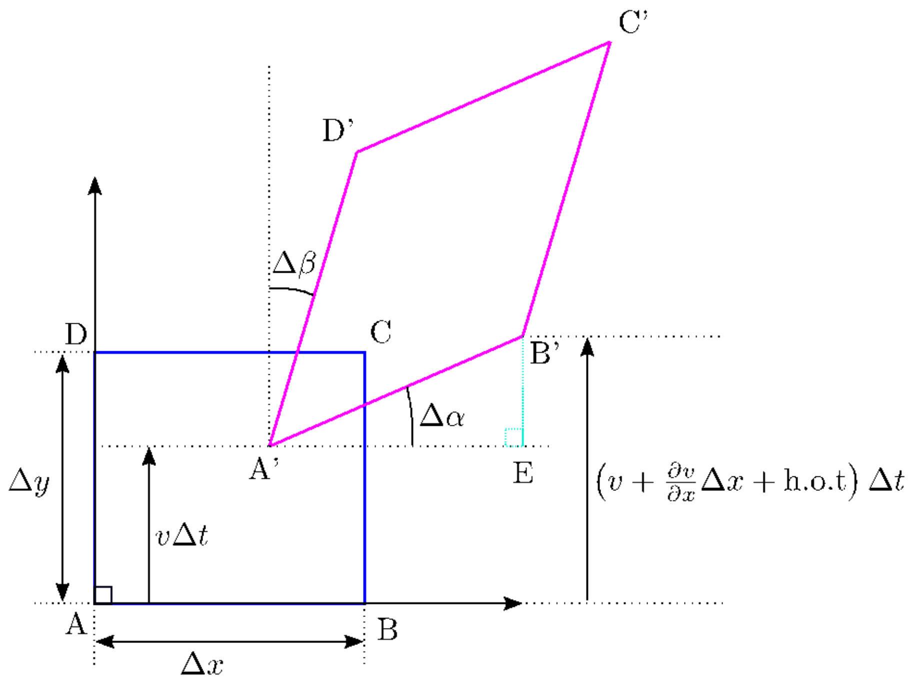

Fig 1. A two-dimensional fluid element ABCD that is rectangular deforms to A'B'C'D' after time interval Δt .

Therefore, to understand angular deformation, we consider a two-dimensional fluid element ABCD that undergoes deformation to A'B'C'D' in time Δt (presented in Fig 1). The angular deformation is well defined by the angles, $\Delta\alpha$ and $\Delta\beta$, denoting the angle of tilting of the lines AB and AD to A'B' and A'D' respectively as the fluid element deforms. Note that the angle for the horizontal line AB, i.e. $\Delta\alpha$ is counter-clockwise, and the angle for the the vertical line AD, i.e. $\Delta\beta$ is clockwise. The issue of angular deformation is essentially to quantify the rate at which these angles are changing. We first quantify the angle $\Delta\alpha$ in terms of the line-lengths in Fig 1.

$$\tan\Delta\alpha = \frac{\mathrm{B'E}}{\mathrm{A'E}}.$$

For a small enough duration, the velocity of the fluid can be approximated to be constant over this time duration. Therefore, the displacement of A in the vertical direction as it moves to A' is $v\Delta t$, and the displacement of B in the vertical direction as it moves to B' is $\left(v + \frac{\partial v}{\partial x}\Delta x + \text{h.o.t.}\right)\Delta t$, where v denotes the y-component of velocity. These displacements are illustrated in Fig 1. Therefore, B'E equals $\left(\frac{\partial v}{\partial x}\Delta x + \text{h.o.t.}\right)\Delta t$, which is approximately equal to $\frac{\partial v}{\partial x}\Delta t$ for a fluid element with small Δx. Similarly, A'E equals $\Delta x + \text{H.O.T}$, which is approximately equal to Δx for small Δt. Using this, $\dot{\alpha} = \lim_{\Delta t \to 0}\frac{\Delta\alpha}{\Delta t} = \frac{\partial v}{\partial x}$. Similarly $\dot{\beta} = \lim_{\Delta t \to 0}\frac{\Delta\beta}{\Delta t} = \frac{\partial u}{\partial y}$.

Shear Deformation and Vorticity: We will now define the rate of shear deformation. As a side note, it is advisable to distinguish shear deformation from linear deformation (or volumetric deformation), which is another type of deformation fluid undergo. Rate of shear deformation is defined as the rate of change of the angle between 2 line elements that were originally perpendicular to each other, which are AB and AD in the context of Fig 1. Examining Fig 1, the change in angle is $(\Delta\alpha + \Delta\beta) - \frac{\pi}{2}$, and so, rate of the change of angle is $(\dot{\alpha} + \dot{\beta})$. Furthermore, since fluid doesn't rotate line a rigid body, and different line elements of the fluid are rotating differently, we define the fluid's angular velocity as the average of angular velocity of two initially perperdicular line elements. Taking AB and AD as these two line elements, and recognizing that $\Delta\alpha$ is counter-clockwise and $\Delta\beta$ is clockwise, angular velocity of fluid is defined as $\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \frac{1}{2}\nabla \mathrm{x} \vec{v}$, which is strictly an artificial definition. In other words, $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Simiarly, the other two components of fluid's angular acceleration are, $\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$ and $\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$. Put together, the vectorial expression for angular velocity is

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} = \frac{1}{2}\nabla \mathrm{x} \vec{v}$$

The curl of the velocity vector, $\nabla \mathrm{x} \vec{v}$, has been part of the formalism of fluid mechanics for traditionally, and is called vorticity. Physically, vorticity represents the rotationality in the flow.

If there is no rotation in a particular flow, then $\nabla \mathrm{x} \vec{v}$ is zero and the flow is called as irrotational flow, otherwise, $\nabla \mathrm{x} \vec{v}$ is non-zero and the flow is called as rotational flow.

Circulation: Circulation is a closely related concept to vorticity. It is defined as $\Gamma = \int_C \vec{v} \cdot d\vec{l}$, Γ is the circulation, and the integral is carried out over the closed contour C along the perimeter elements $d\vec{l}$. The standard tradition is to take the elemental perimeter steps in the counter-clockwise direction. Using the Stokes' theorem in vector calculus, the integral is converted to $\Delta\theta$, where A is the area enclosed by the closed contour C, and $\hat{n}$ is the unit normal to the element dA. This expression qualitatively implies that circulation per unit area is vorticity.

II. Forced Vortex and Free Vortex

To further illustrate the concept of vorticity and flow rotationality/irrotationality, we consider the example of vortices. Mathematically, vortex flow is a type of flow where in the polar co-ordinate system, i.e. $r-\theta$ system, v_θ is the finite velocity component and $v_r = 0$. If the expression for v_θ is $v_\theta = cr$, the vortex is called a Forced Vortex. The flow in a forced vortex is a rigid body like rotation of fluid. On the other hand, if the expression for v_θ is $v_\theta = \frac{c}{r}$, the vortex is called a Free Vortex. Free vortex has irrotoational flow.

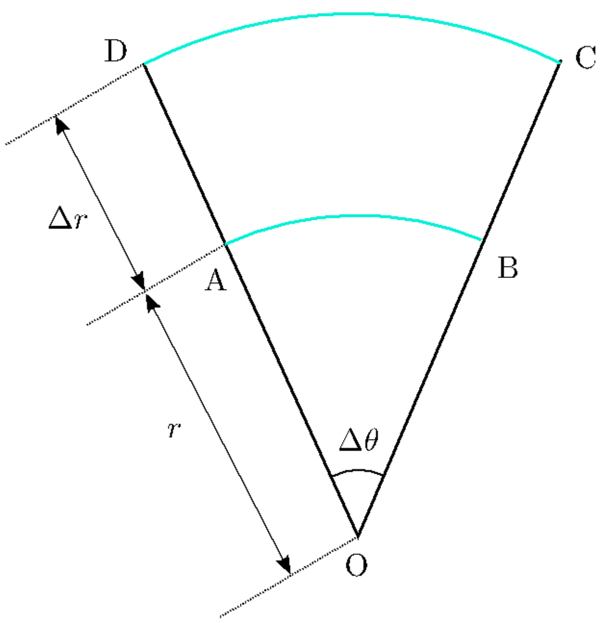

Fig 2. A closed contour ABCD on the arc of angle $\Delta\theta$ to obtain the circulation for vortex flow.

To obtain the circulation for either vortex, we consider a closed contour ABCD on the arc of angle $\Delta\theta$, shown in Fig 2. The integral $\int_C \vec{v} \cdot d\vec{l}$ gets split into four parts, along AB, along BC, along CD, and along DA. Of these, the second and the fourth integrals are zero because for BC and DA, the velocity is perpendicular to $d\vec{l}$. The integral along AB is $-v_\theta(r)\cdot r\Delta\theta$ and the integral along CD is $v_\theta(r+\Delta r)\cdot(r+\Delta r)\Delta\theta$. Thus, the circulation is $(v_\theta(r+\Delta r)\cdot(r+\Delta r)-v_\theta(r)\cdot r)\Delta\theta$.

Substituting the expressions for v_θ, the circulation for Forced vortex is $2cr\Delta r\Delta\theta + c\Delta r^2\Delta\theta$. For a small enough contour ABCD, this approximates to $2cr\Delta r\Delta\theta$, and dividing by area, the vorticity is $2c$. On the other hand, the circulation for the free vortex is $c\Delta\theta\left(\frac{r+\Delta r}{r+\Delta r}-\frac{r}{r}\right)$, which is zero, as expected for an irrotational flow. Vorticity is also zero.

However, pertaining to the free vortex, there is a tricky aspect to this consideration. Let us say there is one chooses the contour OBA. With this countour, the piecewise integral will yield $c\Delta\theta$ as the circulation, which is not 0. This gives a false impression that the flow has some rotationality. The reason for this erroneously obtained anomaly is that the contour for the integral includes O, which is a singular point, i.e. the point with velocity tending to infinity. From this, we deduce that an important rule to follow when integrating over a closed contour to obtain the circulation is that the contour should not include any singular points.

In practice, the forced vortex or the free vortex as standalone models are rarely sufficient to describe an actual practical problem. So, a practical problem that requires considering vortices is often described as a combination of force vortex and free vortex. A common and widely used example is the tornado. A tornado is described as a forced vortex starting from its eye upto a certain radius, beyond which it is described as a free vortex. This model is commonly called as Rankine Vortex.

In the subsequent lecture, we will study linear and volumetric deformation.