Advanced Concepts In Fluid Mechanics

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Lecture – 28
Potential Flow

In the fluid mechanics course we have discussed about the viscous flows. Viscous flows are very important practical considerations in fluid mechanics because we know that every fluid has a viscosity. So, viscous flows are practical representatives of the flows which are taking place all around. Now on a different and idealized side of the paradigm, there is also another type of flow which is called as the ideal fluid flow.

So the question arises about the ideal type of fluid flow. First of all, the ideal type of fluid flow is inviscid which means that it does not have any viscous effect. In this context one needs to keep in mind that inviscid flow does not literally mean that the viscosity of the fluid under consideration is equal to zero. There can be cases when the fluid has a non-zero viscosity but the rate of deformation itself is such that the shear stress (which is the product of the fluid viscosity and the rate of deformation) vanishes. So, inviscid flow means that there is no viscous effect.

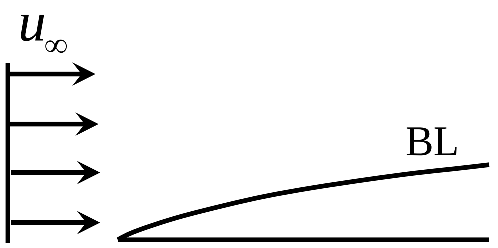

Figure 1. Snapshot of a uniform velocity profile at the entry representing an inviscid flow. The viscous effect exists within the boundary layer (BL).

Let us consider a solid boundary on which a uniform velocity profile is coming. In that case, at least this snapshot of the uniform velocity profile in figure 1 is clearly a depiction of zero shear stress. The reason is that the shear stress is the product of fluid viscosity and the velocity gradient. Since the flow is uniform, there is no velocity gradient which means that the shear stress is equal to zero. So, to that extent, this velocity profile snapshot is an inviscid flow. Now the question arises that whether the entire flow can be treated as an inviscid flow or not given that the velocity profile snapshot at the entry is an inviscid flow. This is a very important question because often we do not answer this question. We see an inviscid flow at some location and then we try to make a speculation from that. We consider that we can treat the entire flow all the time at all places as inviscid one. Now we need to recall the factors (which were discussed in the earlier chapters) which can change the pattern of an inviscid flow. Although the velocity of the fluid is uniform at the entry, we have viscous effects playing a significant role in the entire part of the domain. Irrespective of whether it is the full part of the domain or a part of the domain which is close to the solid boundary (it actually depends on several factors), there is the viscous effect factor. Now the inviscid flow does not have a velocity gradient and therefore, it is also an irrotational flow. This irrotational flow is the keyword which is of more importance to the topic that we are going to study now rather than the inviscid flow. The reason is this effect is something on which we are banking on. So we start with uniform velocity profile at the entry which is clearly irrotational. Now we recapitulate about the rotational and irrotational flow for ease of understanding. The rotationality in the flow is described by the curl of the velocity vector which is also called as vorticity, i.e. $\nabla \times \vec{V} = \text{vorticity}$. $\nabla \times \vec{V} = \text{vorticity}$ is the measure of the strength of rotation in the flow. If it turns out to be a null vector then it means that there is no rotationality in the flow and we call it as irrotational flow. So the present example of uniform velocity profile is a typical example which we can treat as both inviscid flow as well as irrotational flow. It is inviscid flow because there is no shear stress but for our consideration in this particular chapter it is also important to recognize that the curl of the velocity vector $\nabla \times \vec{V}$ is the null vector which means that it is an irrotational flow. Now the question arises that whether we can treat it as inviscid everywhere and irrotational everywhere or not. Clearly, we cannot treat it as inviscid everywhere because of the presence of the solid boundary. The presence of the solid boundary creates a disturbance or perturbation in the momentum and there will be a propagation of this momentum disturbance of the solid boundary towards the outer fluid. That propagation will take place through the fluidic property viscosity. So, if the fluid does not have a zero viscosity, there will be a viscous effect (no doubt about that). The viscous effects will be confined within a thin layer if the Reynolds number is large otherwise the viscous effect will be penetrating deep. For the time being, let us assume that the flow is a high Reynolds number flow. So if it is a high Reynolds number flow, the layer will be thin within which the viscous effect exists; it is called as the boundary layer (as shown in figure 1). So, within this boundary layer the viscous effect is important and outside the boundary layer the viscous effect is not important. It means that outside the boundary layer we can treat the flow as inviscid flow. So, if we can treat the flow as an inviscid flow, then the outer velocity profile which was irrotational will

remain irrotational forever. So the bottom line of our discussion is that if the flow is initially irrotational, then it can be treated as irrotational at other places provided the viscous effects are absent and of course, other body forces are conservative. So, with this consideration we can say that it is not only inviscid or not only irrotational but a combination of these two factors (it is also the combination we are looking for). If it is irrotational at the entry it will remain irrotational at all other places where it is inviscid also. So the combination of inviscid and irrotational flow means that it will remain irrotational at all places. This combination results in a null vector, i.e. $\nabla\times\vec{V}$ is a null vector. If the curl of the velocity vector $\nabla\times\vec{V}$ is a null vector we can express the velocity vector $\vec{V}$ as a gradient of a scalar potential ϕ. This is based on a vector identity which tells that the curl of a gradient of a scalar potential is a null vector $\nabla\times(\nabla\phi)=\vec{0}$. So we can say that $\vec{V}$ and $\nabla\phi$ are synonymous when the right hand side is equal to a null vector. This potential function ϕ is called as the velocity potential. The flows where the velocity potential exists are called as the potential flows. Now the question arises about the reason of studying such an idealized paradigm. This is very important; we will do very interesting mathematics over this chapter of potential flow (which we are starting at this moment) but it is important to recognize the importance of studying this potential flow. When we start discussing the potential flow, we immediately think that this corresponds to the situation when the viscous effect is not important and all practical flows are viscous flows. So the representation of the velocity vector in terms of the scalar potential function may seem like a purely mathematical treatment and therefore question arises about whether there is any practicality towards this or not. Now we try to answer this question. We have considered here high Reynolds number flow; we have a region demarcated as the boundary layer which is adjacent to the solid boundary. The region which is outside the boundary layer, the flow is inviscid and irrotational because we have already imposed an irrotational flow at the entry. In the inside of the boundary layer the flow is definitely characterized by the viscous effect. The common link that relates the outer low with the inner flow is the pressure gradient. There is a pressure gradient which is imposed on the fluid in the boundary layer is the same pressure gradient that is imposed externally. This, of course, is detailed in the chapter of boundary layer theory. This is not within the purview of the present chapter but this is a very important consideration. To know about this pressure gradient within the boundary layer, we need to solve the velocity within the boundary layer. This pressure gradient will be same as the pressure gradient which is imposed in the outer flow. The pressure gradient which is imposed in the outer flow can be obtained by looking into the solution of the velocity in the outer flow which is an inviscid and irrotational flow. Therefore, analysis of inviscid and irrotational flow is required to get the velocity distribution and pressure distribution outside the boundary layer. That pressure distribution outside the boundary layer is imposed on the boundary layer to get the wall shear stress. Therefore, although the fluid does not have a zero viscosity, but an idealized paradigm of inviscid and irrotational flow outside the boundary layer actually helps us to solve the boundary layer equations. This makes it a very practical proposition for doing fluid mechanics analysis.

So, it is not a question of whether the fluid has zero viscosity or the flow is irrotational everywhere or not. Within the boundary layer the flow is clearly rotational. We have to be very careful about the usage of the terms like inviscid and irrotational flow. Whether this consideration of inviscid and irrotational flow can be applied in the entire domain or in a part of the domain is not of much concern. The important thing is that this consideration simplifies the governing equations considerably. With this background, we will start with this chapter of potential flow which is essentially inviscid as well as irrotational flow. By the name of potential flow it is quite clear that it is called as the potential flow because the velocity potential exists. The velocity potential exists for irrotational flow. Now we will make another very important assumption which is the consideration of two-dimensional and incompressible flow. So we consider the two-dimensional and incompressible flow in addition to the consideration of inviscid and irrotational flow. So, the pure kinematic constraint of the incompressibility of the flow tells that $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ where u and v are the velocity components. We can write ψ as $u=\frac{\partial \psi}{\partial y}$, $v=-\frac{\partial \psi}{\partial x}$ like the potential function ϕ; it is another parametric function which is the function of x and y in this case. So the way we define ψ here is just to satisfy the incompressibility condition in a parametric form. When we write ψ in this way, ψ is called as the stream function. In our kinematics chapter we have already discussed that the stream function is constant along a streamline. But the difference between the concept of stream function and streamline is that streamline does not care whether the flow is two-dimensional, incompressible or compressible; streamlines are defined for all types of flow. Only for a two-dimensional and incompressible flow there is a connection between streamline and stream function and the connection is that the stream function is constant along a streamline. We recall the equation of streamline which is given by $\frac{dx}{u} = \frac{dy}{v}$ (here we are not considering the *w*-component of velocity since the flow is a two-dimensional flow). So, ψ becomes function of x and y and we can write $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$. Using the definitions of $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, we get $d\psi = -v\,dx + u\,dy$. Now, along a streamline, $d\psi$ is equal to zero because along a streamline $v\,dx$ is equal to $u\,dy$. So these two terms $-v\,dx$ and $u\,dy$ cancel along a streamline. We can write from the equation of streamline $\left.\frac{dy}{dx}\right|_{\psi = \text{const.}} = \frac{v}{u}$ (so this represents $\psi = \text{constant}$ lines. Similarly we can think of $\phi = \text{constant}$ lines which are called as the equipotential lines and these are very important for potential flow. We have considered a two-dimensional and irrotational flow. Irrotational flows are very general and they can be three-dimensional also. But since we are bringing the stream function and the velocity potential in the same platform we are considering two-dimensional flows. Similar to stream function ψ, the velocity potential ϕ is also function of *x* and *y*. So, $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$; we recall the definition of the velocity potential which is given by $\vec{V} = \nabla \phi$. It also means that $u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$; we substitute these expressions of velocities in the expression of $d\phi$ and we get $d\phi = u\,dx + v\,dy$. So, $\phi = \text{constant}$ means $d\phi$ will be equal to zero which implies that $\left.\frac{dy}{dx}\right|_{\phi = \text{const.}} = -\frac{u}{v}$. Now we multiply the expression of the derivatives representing $\psi = \text{constant}$ lines and $\phi = \text{constant}$ lines and we get $\left.\frac{dy}{dx}\right|_{\psi = \text{const.}} \times \left.\frac{dy}{dx}\right|_{\phi = \text{const.}} = -1$ provided *u* and *v* are not zero. If the magnitude of any of the two velocities is equal to zero then there is a chance of division by zero and the entire process of multiplication breaks down. This singularity point at which the velocity *u* or *v* (or both) are equal to zero is known as the stagnation point. So stagnation point is the point at which the velocities are equal to zero which means that the fluid stagnates. So we can infer from here that $\psi = \text{constant}$ lines and $\phi = \text{constant}$ lines are orthogonal to each other everywhere in the flow field except at the stagnation point. Since these lines are orthogonal to each other, we can define a complex function based on ψ and ϕ. Now we will define this complex function.

Let F be a complex function in the form $F = \phi + i\psi$. In this context, we recall the definition of a complex number z which is represented in the form $z = x + i\,y$. The basic premise on the basis of which we can write this form is that *x* and *y* are in orthogonal direction. In the present case, ψ and ϕ are in orthogonal direction. Therefore, except the consideration at the stagnation point, we can define a complex function as $F = \phi + i\psi$. The advantage of using this complex function is that we can now treat the velocity potential and the stream function together by a single complex variable. ψ and ϕ are functions of *x* and *y*. Using the form $F = \phi + i\psi$ we can write $\frac{dF}{dz} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z}$ (this is clear because it is functions of two variables *x* and *y*). Now $\frac{\partial F}{\partial x}$ is equal to $\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$; $\frac{\partial x}{\partial z}$ is equal to 1; $\frac{\partial F}{\partial y}$ is equal to $\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}$ and $\frac{\partial y}{\partial z}$ is equal to $\frac{1}{i} = -i$ (because $i^2 = -1$). Substituting these expressions in the derivative $\frac{dF}{dz}$ we get, $\frac{dF}{dz} = \left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right)\times 1 + \left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right)\times(-i)$. Now the question arises about whether the function *F* has a defined $\frac{dF}{dz}$ or not. If it does not have a defined $\frac{dF}{dz}$ we can say that it is not complex differentiable or in another terminology it is not analytic. So the terms analytic and complex differentiable are two synonymous terms. Now we have to check whether this complex derivative $\frac{dF}{dz}$ exists or not. We have to check this by noting that whether this derivative is independent of the direction in which the derivative is calculated. The derivative $\frac{dF}{dz}$ along $x = \text{constant}$ is given by $\left.\frac{dF}{dz}\right|_{x=\text{const}} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$.

Similarly, the derivative $\frac{dF}{dz}$ along $y = \text{constant}$ is given by $\left.\frac{dF}{dz}\right|_{y=\text{const}} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$.

Now the question arises about whether these two derivatives are equal to each other or not. We recall from our previous discussion that $u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$ and $v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$ (this was the definition of the stream function ψ and the velocity potential ϕ through velocities *u* and *v*). These conditions are called as Cauchy-Riemann conditions. In the present case the Cauchy-Riemann conditions are satisfied since we have considered two-dimensional, incompressible and irrotational flow. Using these forms of *u* and *v*, we get $\left.\frac{dF}{dz}\right|_{x=\text{const}} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} = u - iv$ and $\left.\frac{dF}{dz}\right|_{y=\text{const}} = u - iv$, so, both the derivatives are equal to $u - iv$ provided that the Cauchy-Riemann conditions are satisfied. Before the application of the Cauchy-Riemann conditions, the mathematics part does not understand about the flow; it was just the rule of partial derivatives. But it eventually understands that the flow is two-dimensional, incompressible and irrotational flow through invoking the Cauchy-Riemann conditions. Only then the function *F* becomes analytic or complex differentiable. So, it implies that the derivatives $\left.\frac{dF}{dz}\right|_{x=\text{const}}$ and $\left.\frac{dF}{dz}\right|_{y=\text{const}}$ are the same where *x* and *y* are the two representative orthogonal directions. We can say that the function *F* is analytic or complex differentiable which means that the complex derivative exists in the complex plane.

Overall, in the present chapter, we have discussed about the importance of the potential flow. We have introduced the way of defining a complex potential $F = \phi + i\psi$ whose real part is the velocity potential (ϕ) and imaginary part is the stream function (ψ). In the subsequent chapters we will use this definition to bring out certain important characteristics of potential flow through defining first very simple types of flow and then linearly superimposing the simple flows to get more complex flows.