Advanced Concepts In Fluid Mechanics
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Lecture - 27
Application of Momentum Integral Method and Boundary Layer Separation

In the previous chapter we discussed about the momentum integral equation. In the present chapter we will discuss about how this equation can be applied to calculate various engineering parameters. The momentum integral equation for the flow over a flat plate is given by $\frac{\tau_w}{\rho u_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$. Let us assume that the velocity ratio $\frac{u}{u_\infty}$ is a cubic polynomial, i.e. $\frac{u}{u_\infty} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3$. This is an approximate form. We have to obtain the coefficients a_0, a_1, a_2 and a_3 from the known matching conditions. We will write the conditions according to the priority because had we considered a linear polynomial (i.e. a straight line), then only a_0 and a_1 had to be determined. So in that case only two conditions would be required and question arises about which two conditions we need to use. To do this we write the most prioritized boundary conditions. The first of the two boundary conditions that must be satisfied is at $y = 0$, $u = 0$ which is the no-slip boundary condition. The second boundary condition is at $y \to \infty$, $u = u_\infty$. $y \to \infty$ is physically replaced by $y = \delta$ because at $y = \delta$, the velocity u becomes equal to 99% of u_∞; so at $y = \delta$, $u = u_\infty$. The third boundary condition is at $y = \delta$, $\frac{\partial u}{\partial y} = 0$. These three boundary conditions are very obvious. However, here we have four coefficients in the approximation of $\frac{u}{u_\infty}$; so need a fourth boundary condition. For getting the fourth boundary condition, we should look into the boundary layer equation at the wall. The boundary layer equation is given by $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$. Now u is equal to zero at the wall because of no-slip condition, v is equal to zero at the wall because of no-penetration condition; so, $\frac{\partial^2 u}{\partial y^2}$ must be equal to zero at the wall. So, the fourth boundary condition is at $y=0$, $\frac{\partial^2 u}{\partial y^2}=0$. With these conditions we can calculate the coefficients a_0, a_1, a_2 and a_3. Now we substitute these boundary conditions to get the final expression of $\frac{u}{u_\infty}$; this is a very trivial algebra. Substituting the first two boundary conditions (i.e. at $y=0$, $u=0$ and at $y=\delta$, $u=u_\infty$), we get the expressions $a_0=0$ and $a_0+a_1+a_2+a_3=1$. Now we calculate the first and second derivatives of velocity which becomes $\frac{\partial u}{\partial y}=u_\infty\left[\frac{a_1}{\delta}+\frac{2a_2 y}{\delta^2}+\frac{3a_3 y^2}{\delta^3}\right]$ and $\frac{\partial^2 u}{\partial y^2}=u_\infty\left[\frac{2a_2}{\delta^2}+\frac{6a_3 y}{\delta^3}\right]$ respectively. Using the expressions of $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ and the two boundary conditions $y=\delta$, $\frac{\partial u}{\partial y}=0$ and $y=0$, $\frac{\partial^2 u}{\partial y^2}=0$; we get the equations $a_1+2a_2+3a_3=0$ and $a_2=0$ respectively. Now we have $a_0=0$ and $a_2=0$; we get two simplified equations $a_1+a_3=1$ and $a_1+3a_3=0$ from which we get $a_1=\frac{3}{2}$ and $a_3=-\frac{1}{2}$. The final form of the velocity profile becomes $\frac{u}{u_\infty}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$. Now we substitute this expression of the velocity profile $\frac{u}{u_\infty}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$ in the momentum integral equation $\frac{\tau_w}{\rho u_\infty^2}=\frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$ and we get the following expression $\frac{\tau_w}{\rho u_\infty^2}=\frac{d}{dx}\int_0^\delta \left\{\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\left[1-\frac{3}{2}\frac{y}{\delta}+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy$. This is just a polynomial integration so we do not need any big trick here. There is a good systematic way of doing this, i.e. to substitute $\frac{y}{\delta}=\eta$ as a variable which results $\frac{\tau_w}{\rho u_\infty^2}=\frac{d}{dx}\int_0^\delta \left\{\frac{3}{2}\eta-\frac{1}{2}\eta^3\right\}\left[1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right]dy$. Taking a differential on both sides of $\frac{y}{\delta}=\eta$, we get $dy=\delta\, d\eta$. Now we look into the wall shear stress (τ_w) which depends on the velocity gradient $\frac{\partial u}{\partial y}$. From the expression

$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ we get $\frac{\partial u}{\partial y} = u_\infty \left[\frac{3}{2\delta} - \frac{3y^2}{2\delta^3}\right]$. The wall shear stress τ_w is given by $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{3\mu u_\infty}{2\delta}$. Substituting this expression of the wall shear stress τ_w in the momentum integral equation $\frac{\tau_w}{\rho u_\infty^2} = \frac{d}{dx}\int_0^\delta \left\{\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy$, we get $\frac{\frac{3\mu u_\infty}{2\delta}}{\rho u_\infty^2} = \frac{d\delta}{dx}\int_0^1 \left\{\frac{3}{2}\eta - \frac{1}{2}\eta^3\right\}\left[1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right] d\eta$ where the substitutions of $\frac{y}{\delta} = \eta$, $dy = \delta\, d\eta$ and the changing of limits of integral have been taken into account. The integral $\int_0^1 \left\{\frac{3}{2}\eta - \frac{1}{2}\eta^3\right\}\left[1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right] d\eta$ will give a numerical value; let us consider this to be equal to *A*, so, $\frac{3\mu u_\infty}{2\delta\rho u_\infty^2} = A\frac{d\delta}{dx}$. One u_∞ will get canceled from the numerator and the denominator, $\frac{\mu}{\rho}$ is replaced by the kinematic viscosity $\nu = \frac{\mu}{\rho}$ and the relationship becomes $\delta\, d\delta = \frac{3}{2}\frac{\nu}{A\, u_\infty} dx$. If we integrate both sides of this, we get $\frac{\delta^2}{2} = \frac{3}{2}\frac{\nu}{A\, u_\infty} x + C$ where *C* is an integration constant. To obtain this constant we apply the boundary conditions at $x \to 0, \delta \to 0$ which is at the edge of the boundary layer. Substituting this boundary condition, the value of the constant *C* is found to be equal to zero. So we get, $\delta^2 = \frac{3\nu x}{A\, u_\infty}$ and $\frac{\delta}{x} = \sqrt{\frac{3}{A}}\sqrt{\frac{\nu}{u_\infty x}} = \sqrt{\frac{3}{A}}\,\mathrm{Re}_x^{-1/2}$. So, again, we get the same thing, i.e. $\frac{\delta}{x}$ scales with $\mathrm{Re}_x^{-1/2}$. If we calculate $\sqrt{\frac{3}{A}}$ we will get it to be equal to 4.64 (one can check this calculation by going through the algebra to see whether it is equal to 4.64 or not). In Blasius equation we got $\frac{\delta}{x} = 5\,\mathrm{Re}_x^{-1/2}$ and here we get $\frac{\delta}{x} = 4.64\,\mathrm{Re}_x^{-1/2}$ which is not that bad considering that this is an approximate solution. Whatever be this approximation, it will be further less if we calculate the skin friction coefficient and the drag coefficient. The method of calculation of the skin friction coefficient and the drag coefficient is exactly the same as we followed in the similarity solution, so this is left here as an exercise. The only difference is that here we have to use the velocity profile $\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ instead of the velocity profile that we got from the similarity solution. Also, δ is a function of *x* for which the relationship $\frac{\delta}{x} = 4.64 \mathrm{Re}_x^{-1/2}$ needs to be substituted here. Then it will complete the velocity distribution from which one can further obtain different parameters like the skin friction coefficient, the drag coefficient. Since, the skin friction coefficient C_{fx} is defined as $\frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$, it can be calculated by simply evaluating the integral $\frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$, substituting relationship of δ and *x* and then multiplying it by 2. Upon integrating we will get the dependence of the skin friction coefficient C_{fx} on the Reynolds number. Finally we will see that the skin friction coefficient C_{fx} becoming scaled with $\mathrm{Re}_x^{-1/2}$, i.e. $C_{fx} \sim \mathrm{Re}_x^{-1/2}$ (where Re_x is the local Reynolds number) and the drag coefficient C_D becoming scaled with $\mathrm{Re}_L^{-1/2}$, i.e. $C_D \sim \mathrm{Re}_L^{-1/2}$ (where Re_L is the Reynolds number based on the length scale *L*).

The next topic in the remaining chapter, which we will cover, is a very important concept. So far we have discussed the growth of boundary layer without the presence of pressure gradient. Therefore, question arises about what happens if there is a boundary layer in presence of pressure gradient. This is a very interesting topic because if we understand it properly, we will get a clue of the motion of various sports balls and the effect of the boundary layer growth on their motion.

Let us imagine that there is a sphere or a cylinder. There is a free stream with u_∞ falling on this object. The streamlines which are far away from the cylinder or the sphere will not understand the effect of this body. So the streamline far away from the body will be straight and parallel to the horizontal axis. With respect to this as a streamline and the body of the cylinder as a streamline, this region actually represents a converging flow passage (as shown in figure 1). Since it is a converging flow passage, the area of the flow is decreasing and we have the fluid velocity accelerating. Now question arises about the factor which is accelerating the flow. It is the driving pressure gradient which is

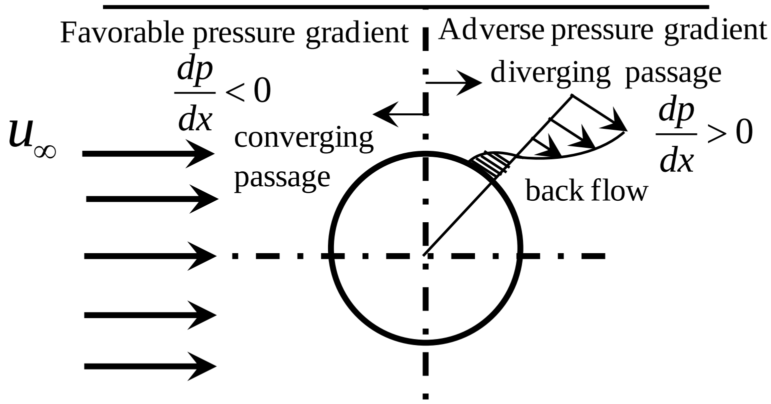

Figure 1. Pictorial depiction of regions of favorable pressure gradient ($\frac{dp}{dx} < 0$) and adverse pressure gradient ($\frac{dp}{dx} > 0$). We have a sphere and there is a uniform flow u_∞ which is falling on this object.

accelerating the flow. The reason is that if we imagine the far stream outside the boundary layer, the viscous effects are not important there. So it is the driving pressure gradient which is actually accelerating the flow. That pressure gradient which is active till this region is called as the favorable pressure gradient because it is favorable to driving the flow. Favorable pressure gradient is given by $\frac{dp}{dx} < 0$. So this favorable pressure gradient part is on the left half portion of the diagram. On the other side (i.e. between the outside streamline and the body of the object as streamline on the right half of the diagram), the flow passage is diverging. When the flow passage is diverging, the velocity is anyway reducing because of this. So, there is a decelerating effect and that decelerating effect is due to a pressure gradient which is slowing the fluid down. This is called as adverse pressure gradient which is given by $\frac{dp}{dx} > 0$. The basic difference between the regions on the left half portion and right half portion is as follows. In the left half portion, the fluid will anyway accelerate in the outer stream; in the inner stream it will be slowed down by viscosity. But the favorable pressure gradient outside the boundary layer will try to drag the fluid. So there will be a monotonous growth of the boundary layer. However, in the region on the right half portion of the diagram, the inertia of the fluid which the fluid gained in the previous region; will still try to drive the fluid. Here, the pressure gradient is opposing the fluid in the bulk and at the wall; the viscous effect is also opposing the fluid. So the fluid has only its inertia to support, but the pressure gradient is opposing it and the viscous force at the wall is also opposing it. It comes to a condition that the fluid cannot sustain its forward momentum anymore. When the fluid cannot sustain its forward momentum anymore there can be a local backflow. So, if there is a local back flow close to the wall, we say that the boundary layer has separated. The reason is that this point (as shown in figure 1) is artificially a point of zero velocity as if the solid boundary is shifted to make sure that there is a monotonic growth in the boundary layer. Actually, there is no monotonic growth in this region and there is a low pressure region which is created at the back of this after boundary layer separation. As we have told earlier, because there is a boundary layer separation, we cannot use the boundary layer theory because there is no monotonous growth of boundary layer. So, for boundary layer separation, adverse pressure gradient is required but adverse pressure gradient does not necessarily mean that there will be boundary layer separation. So, adverse pressure gradient is a necessary condition for boundary layer separation but not a sufficient condition. The reason is that, despite having adverse pressure gradient, the fluid might be still having some inertia in order to maintain the forward motion. But having adverse pressure gradient is a necessary condition because without that the boundary layer separation will not take place.

With this physical understanding of the role of the adverse pressure gradient, when the boundary layer has separated or get detached from the wall, the consequence is that in the region on the right hand side there is low pressure and in the region on the left hand side there is high pressure. The reason is that there is a driving pressure gradient and because of this, there is a pressure distribution across the body which is now no more symmetric. Because of this pressure distribution, there is a drag force which is called as the form drag or the pressure drag. The reason of calling it as form drag or pressure drag is that, this depends on the geometric form of the body. Also it is a function of the pressure distribution on the body. That is why it is called as form drag or pressure drag. By this time, we have learnt about two important sources of drag forces; one is the skin friction drag while the other one is the form drag or pressure drag. The net drag is the resultant consequence of skin friction drag and form drag. Skin friction drag is due to tangential force and form drag is due to normal force. Question arises about how we consider the drag force. Whatever is the resultant force distribution, the component of that force in the direction of the flow is called as drag force. A part of that component will be due to skin friction and a part will be due to form drag or pressure drag. Now we will try to draw the velocity profiles, the velocity gradient profiles and the second derivative of velocity for the cases of $\frac{dp}{dx} < 0$ and $\frac{dp}{dx} > 0$. We assume that for both the cases, there will be no boundary layer separation. For $\frac{dp}{dx} < 0$, anyway there will be no boundary layer separation; for $\frac{dp}{dx} > 0$, we will assume that it is still such that the boundary layer separation has not occurred.

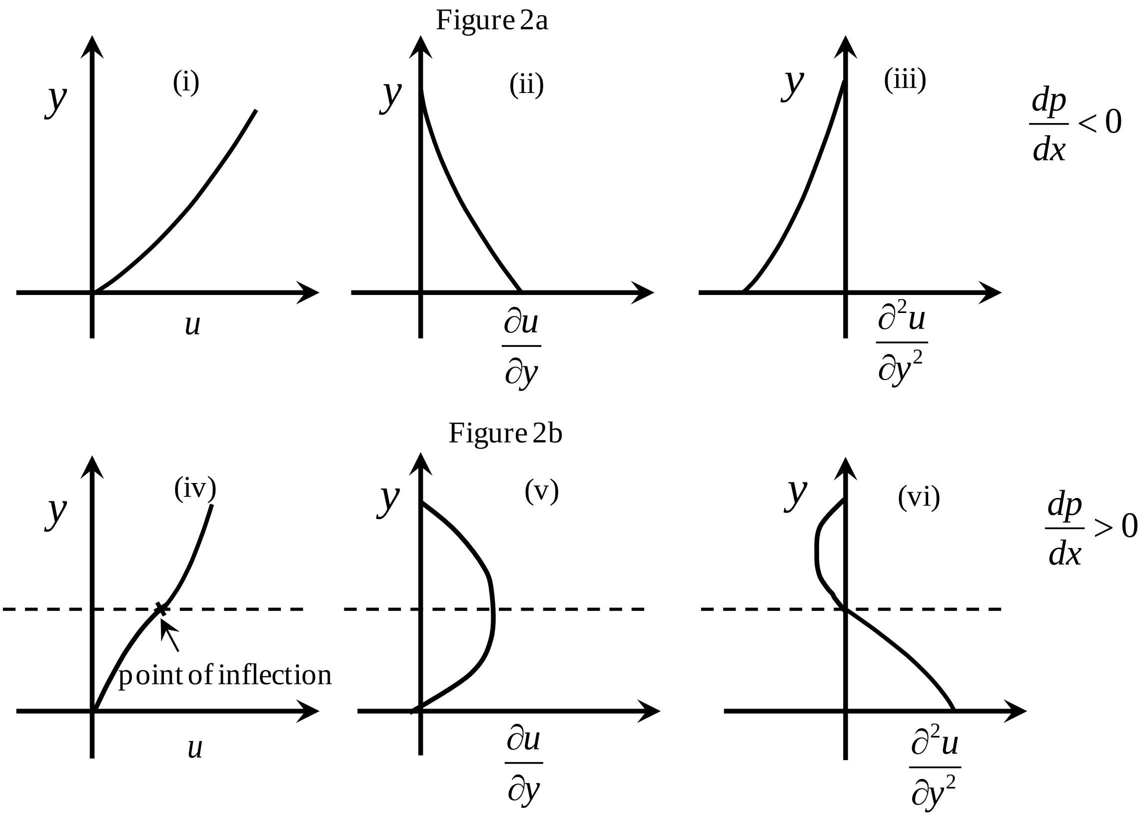

Figure 2. Variations of velocity (u), velocity gradient $\left(\frac{\partial u}{\partial y}\right)$ and second derivative of velocity $\left(\frac{\partial^2 u}{\partial y^2}\right)$ as a function of y. Figure 2a corresponds to $\frac{dp}{dx} < 0$ while figure 2b corresponds to $\frac{dp}{dx} > 0$.

Now we will focus on the plot shown in figure 2. The upper half portion of this figure corresponds to $\frac{dp}{dx} < 0$ while the lower half portion of this figure corresponds to $\frac{dp}{dx} > 0$. Now looking into the entire diagram it is very difficult for us to follow. Therefore, one needs to look into the diagram on the extreme right first (i.e. the variation of second derivative of velocity with y). From the right side, we need to go to the left hand side such that it will be easier to follow the logical sequence in which the diagrams are arrived at. So we will focus on the second derivative of velocity first. Let us write the boundary layer equation which is given by $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{dp}{dx}$. Now, at the wall, the velocity u is zero, the velocity v is also zero. So, at the wall, we get, $\left.\frac{\partial^2 u}{\partial y^2}\right|_w=\frac{1}{\nu}\frac{dp}{dx}$. If $\frac{dp}{dx}$ is negative (which is favorable), then $\frac{\partial^2 u}{\partial y^2}$ is negative at the wall. Now question arises about what happens at the far stream. $\frac{\partial^2 u}{\partial y^2}$ can be written as $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$; $\frac{\partial u}{\partial y}$ decreases with increasing y because it has to zero at the far stream. So, $\frac{\partial^2 u}{\partial y^2}$ will be therefore negative at the far stream. Accordingly, the diagram of $\frac{\partial^2 u}{\partial y^2}$ with y will be something like what is drawn in figure 2a (iii). Now we focus on the first derivative $\frac{\partial u}{\partial y}$. For $\frac{dp}{dx}<0$, there is no boundary layer separation. So the shear stress (represented by $\tau_w=\mu\frac{\partial u}{\partial y}$) will be positive at the wall and $\frac{\partial u}{\partial y}$ will be positive at the wall. $\frac{\partial u}{\partial y}$ will be tending towards zero as we go along the y direction towards infinity (shown in figure 2a (ii)). The variation of u as a function of y will be the usual one as can be seen from figure 2a (i). More interesting case will be in presence of adverse pressure gradient (i.e. $\frac{dp}{dx}>0$) which is pictorially depicted in figure 2b.

For the case of $\frac{dp}{dx}>0$, $\frac{\partial^2 u}{\partial y^2}$ is greater than zero at the wall. Far away from the wall, $\frac{\partial^2 u}{\partial y^2}$ has to be negative; so $\frac{\partial^2 u}{\partial y^2}$ is asymptotically zero somewhere. Since it is negative far away from the wall and positive at the wall, it must have crossed the y axis somewhere since it is a continuous function. The variation of $\frac{\partial^2 u}{\partial y^2}$ in this case is shown in figure 2b (iii). It means that there is a point at which $\frac{\partial^2 u}{\partial y^2}$ will be equal to zero. At the wall, it is positive and far away from the wall, it is negative; so being a continuous function, it must cross 0 somewhere. The location where $\frac{\partial^2 u}{\partial y^2}$ is equal to zero, there $\frac{\partial u}{\partial y}$ must be maximum. The plot of $\frac{\partial u}{\partial y}$ in this case is shown in figure 2b (ii). The point where $\frac{\partial u}{\partial y}$ is maximum we will have a point of inflection in the velocity profile. This point of inflection is marked in the velocity profile shown in figure 2b (i). This point of inflection is a hallmark of the possibility of boundary layer separation.

So by physically looking at the velocity profile, if we can identify a point of inflection, then we may sense that there could be a possibility of boundary layer separation. Overall, we have discussed about the boundary layer in presence of pressure gradient. We began the discussion in absence of the pressure gradient. Now, in presence of pressure gradient, we have identified that there are two types of drag forces, namely, skin friction drag and form drag or pressure drag. Question remains about how these drag forces vary for different types of flows. For example, question appears that if the flow is not laminar and becomes turbulent, then what will be the change in the drag force. We will discuss that part when we will complete a little bit of discussion on turbulence and some introduction to turbulent flows. Then we will come back to this point and discuss that, if we have a turbulent flow instead of a laminar flow, then how there will be the change in the drag force distribution and how it can influence an engineering device and even motion of cricket balls or tennis balls.