Advanced Concepts In Fluid Mechanics
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Lecture – 26
Momentum Integral Method

In the previous chapter, we discussed about the Blasius equation and its implications in terms of calculating the wall shear stress and the drag force. Now question may arise about why the wall shear stress and the drag force are calculated. The reason is that in engineering, wall shear stress and drag force these two are the two most relevant quantities which are utilized for design. The velocity profile is fundamentally important but to an extent since it is needed for the calculation of the drag force. In the present chapter we will continue with that and we will learn about two important definitions.

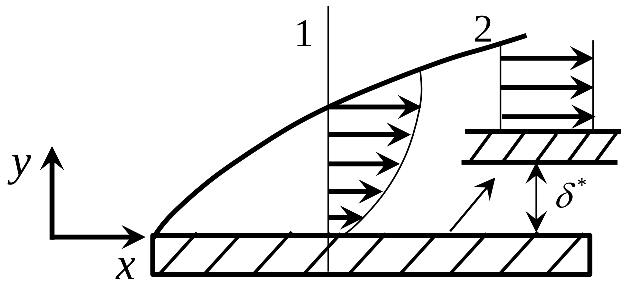

Figure 1. Schematic of the flow over a flat plate where we have the velocity profile in the boundary layer (as shown in section 1). On the right side there is another velocity profile if we imagine that the boundary layer is displaced by thickness δ^*.

First one is the displacement thickness which is denoted by δ^*. To describe the displacement thickness, let us consider the flow over a flat plate as an example where we have a boundary layer and the velocity profile in the boundary layer can be shown by the velocity profile drawn at section 1 of figure 1. Now we imagine that the boundary is physically displaced by a distance and the corresponding velocity profile is drawn at section 2 of figure 1. At section 2 the velocity profile is uniform in nature while the flow rate at section 1 is equal to the flow rate at section 2. The imaginary displacement of the boundary is called as the displacement thickness denoted by the parameter δ^*. So the whole idea is that this imaginary shifting of the boundary may be conceptualized. At section 1, this is viscous flow and at section 2, this is an idealized inviscid flow. If these two flows would give rise to the same flow rate then obviously the boundary has to be displaced for flow at section 2 because at section 1 there is a deficit of flow as evident from the velocity profile. So this deficit in the flow has to be compensated by putting the boundary somewhere above. Now we equate the flow rate at the two sections which is given by $\int_0^\delta u\,dy = \left(\delta - \delta^*\right) u_\infty$. The effective transverse length is at section 2 is equal to $\delta - \delta^*$ since the boundary is being displaced by the length δ^*. Then we can write $\delta^* u_\infty = \int_0^\delta \left(u_\infty - u\right) dy$ where δu_∞ is replaced by $\int_0^\delta u_\infty dy$, i.e. $\int_0^\delta u_\infty dy = \delta u_\infty$. So, the expression of δ^* is given by $\delta^* = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy$. So if we know the velocity profile we can calculate the displacement thickness δ^*. This δ^* may be interpreted in a little bit different way which is normally not discussed in undergraduate text books. We will go through that interpretation.

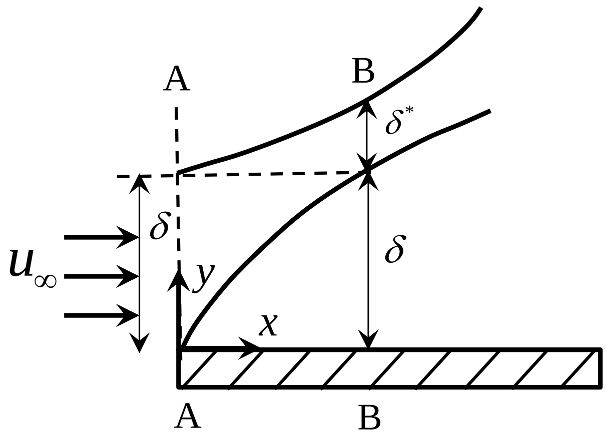

Figure 2. Schematic of the flow over a flat plate where we have the velocity profile in the boundary layer. The imaginary displacement of the boundary layer is denoted by δ^*.

Let us consider the flow over a flat plate as an example and we have a boundary layer as drawn and the corresponding velocity profile is shown in figure 2. Let us consider the thickness of the boundary layer as δ. Now we make an imaginary construction where we draw a line parallel to the plate which will cut the vertical height at the thickness δ. Now we start constructing streamline through this. If we construct streamline through this the streamline will be bent. The reason is the flow rate must be same at the two sections AA and BB. In the section BB the flow is reduced because of boundary layer to compensate for that; so the streamline should be bent upwards to have a region where we have some additional flow. So, the displacement of the streamline is denoted by δ^*. The flow rate is conserved between the two sections AA and BB. The velocity at section AA is u_∞, in section BB inside the boundary layer the velocity is u while outside the boundary layer the velocity is equal to u_∞. So we can write $u_\infty \delta = \int_0^\delta u\,dy + u_\infty \delta^*$. The flow rate across the section AA is equal to the flow rate across section BB including the streaming of course. Within the boundary layer the flow rate is not remaining conserved because the boundary layer is arresting the velocity. So here also we will get $\delta^* = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy$. Interestingly although mass is conserved across these two sections, momentum is not conserved. The momentum transport rate per unit width (or can be written as momentum flux if it is expressed as per unit area) across the section AA = $\rho u_\infty^2 \delta$. Similarly, the momentum flux per unit width across the section BB = $\int_0^\delta \rho u^2 dy + \rho u_\infty^2 \delta^*$. The momentum transport rate is expected to be more across the section AA. So, the difference between momentum fluxes between the section AA and the section BB is = $\rho u_\infty^2 \delta - \int_0^\delta \rho u^2 dy - \rho u_\infty^2 \delta^* = \rho u_\infty^2 \delta - \int_0^\delta \rho u^2 dy - \rho u_\infty^2 \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy$ $= \rho u_\infty^2 \delta - \int_0^\delta \rho u^2 dy - \rho u_\infty^2 \delta + \rho \int_0^\delta u u_\infty dy = -\int_0^\delta \rho u^2 dy + \rho \int_0^\delta u u_\infty dy$. So the difference becomes equal to $\text{difference} = \rho \int_0^\delta \left(u u_\infty - u^2\right) dy$. If we normalize this difference by ρu_∞^2 then it becomes $\frac{\text{difference}}{\rho u_\infty^2} = \int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$ which is called as momentum thickness, just a name given to it. Physically it represents the deficit in the momentum across the two sections over which the flow rate is conserved but the momentum is not conserved.

The momentum thickness or the deficit of momentum is a very important practical parameter because it is related to the wall shear stress. We will show how the momentum thickness relates to the wall shear stress. To show how it relates to the wall shear stress, there is a very elegant method called as the momentum integral method which discusses about that. Now the genesis of the momentum integral method is as follows. The Blasius equation has only a numerical solution and the numerical solution is not very straightforward to obtain considering the resources that were available during the time at which this equation was first introduced. Nowadays it is very easy to solve the Blasius equation but it was not so at the time when it was first introduced. So what was effectively needed was essentially something like an approximate way of solving the boundary layer equation and this is called as momentum integral method. This method was introduced by Von Karman, so it is also known as Von Karman's momentum integral method. The basis of this method is very simple, i.e. to satisfy the boundary layer equations in an integral sense. To do that let us take an example of flow over a flat plate. So we have the boundary layer equation as $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$. We integrate this boundary layer equation with respect to *y* and we get

$$\int_0^\delta u\frac{\partial u}{\partial x}dy+\int_0^\delta v\frac{\partial u}{\partial y}dy=\nu\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy \tag{1}$$

Then we will do some mathematical manipulation. We evaluate the integral $\int_0^\delta v\frac{\partial u}{\partial y}dy$ using integration by parts which results $\int_0^\delta v\frac{\partial u}{\partial y}dy=\left[vu\right]_0^\delta-\int_0^\delta \frac{\partial v}{\partial y}u\,dy$ where *v* is chosen as the first function and $\frac{\partial u}{\partial y}$ as the second function. Assuming two-dimensional incompressible flow we can write $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Using this the integral $\int_0^\delta v\frac{\partial u}{\partial y}dy$ can be written as $\int_0^\delta v\frac{\partial u}{\partial y}dy=\left[vu\right]_0^\delta+\int_0^\delta \frac{\partial u}{\partial x}u\,dy$. So, the left hand side of equation (1) becomes equal to $2\int_0^\delta u\frac{\partial u}{\partial x}dy+\left[vu\right]_0^\delta$. Now $2\int_0^\delta u\frac{\partial u}{\partial x}dy$ can be written as $2\int_0^\delta u\frac{\partial u}{\partial x}dy=\int_0^\delta \frac{\partial}{\partial x}u^2dy$. Now we need to evaluate the expression $\left[vu\right]_0^\delta$ in which u at $y=\delta$ is equal to u_∞ but v at $y=\delta$ is not known. To know this we need to integrate the continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Now integrating this with respect to *y* we get $\int_0^\delta \frac{\partial u}{\partial x}dy+\int_0^\delta \frac{\partial v}{\partial y}dy=0$ in which we will evaluate the term $\int_0^\delta \frac{\partial v}{\partial y}dy$ only. This integral $\int_0^\delta \frac{\partial v}{\partial y}dy$ is equal to $\left[v\right]_0^\delta$ in which *v* at $y=\delta$ is equal to v_∞ while *v* at $y=0$ is equal to zero because of no-penetration boundary condition. So, the integral $\int_0^\delta \frac{\partial v}{\partial y}dy$ becomes equal to v_∞ and $v_\infty = -\int_0^\delta \frac{\partial u}{\partial x} dy$. Using this, the term $\left[vu\right]_0^\delta$ will be equal to $-\int_0^\delta \frac{\partial u}{\partial x} u_\infty \, dy$ since u at $y=\delta$ is equal to u_∞ and both u and v are zero at $y=0$ because of no-slip and no-penetration boundary conditions. The integration $-\int_0^\delta \frac{\partial u}{\partial x} u_\infty \, dy$ can be rewritten as $-\int_0^\delta \frac{\partial}{\partial x}\left(u u_\infty\right) dy$. Using these mathematical rearrangements, the modified from of equation (1) becomes

$$\int_0^\delta \frac{\partial}{\partial x}\left(u^2 - u u_\infty\right) dy = \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} dy \tag{2}$$

The integral on the right hand side of equation (2) becomes equal to $-\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ because at $y=\delta$, u does not vary further with y and therefore, $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$. So, we get $\int_0^\delta \frac{\partial}{\partial x}\left(u^2 - u u_\infty\right) dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$. Now the next important question is that whether we can bring the derivative $\frac{\partial}{\partial x}$ out of the integral or not. The answer is we cannot do this since δ is a function of x. So, for that we have to use the Leibniz's rule which is the rule of differentiation under the integral sign. The Leibniz's rule is given below

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F\left(x,y\right) dy = \int_{a(x)}^{b(x)} \frac{\partial F}{\partial x} dy + F\left(x,b\right)\frac{db}{dx} - F\left(x,a\right)\frac{da}{dx} \tag{3}$$

The derivation of this rule is of course not within the purview of this course but we can see a very important analogy of this rule with the Reynolds transport theorem. $\frac{d}{dx}\int_{a(x)}^{b(x)} F\left(x,y\right) dy$ is like the net rate of change for a system, $\int_{a(x)}^{b(x)} \frac{\partial F}{\partial x} dy$ is the change with respect to the control volume while $F\left(x,b\right)\frac{db}{dx}$ and $F\left(x,a\right)\frac{da}{dx}$ are the outflow and inflow respectively. This is remarkable because Leibniz's rule is a theorem of mathematics which does not understand transport like fluid flow, heat transfer and mass transfer. On the other hand, Reynolds transport theorem is purely based on physical considerations. So we can see how these two concepts merge up to define this rule of differentiation under the integral sign. In the present case, the function $F(x,y)$ is equal to $u^2 - u u_\infty$. Now we need to think about the correction terms $F\left(x,b\right)\frac{db}{dx}$ and $F\left(x,a\right)\frac{da}{dx}$.

In the present case b is equal to δ and $F\left(x,b\right) = {u_\infty}^2 - u_\infty u_\infty = 0$. Since a is equal to zero, $F\left(x,\mathrm{a}\right) = 0 - 0 = 0$. So the correction term becomes zero and we can write

$$\int_0^\delta \frac{\partial}{\partial x}\left(u^2 - u u_\infty\right)dy = \frac{d}{dx}\int_0^\delta \left(u^2 - u u_\infty\right)dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{4}$$

This is a very classical example that ignorance sometimes can be a blessing. If someone does not know that there is a rule called as Leibniz's rule, one will freely take the derivative outside the integral and proceed. In the present case since the correction terms becomes fortunately equal to zero, ignorance is coming out to be a blessing. But if the correction term is non-zero then we cannot take the derivative out of the integral sign and then ignorance can be a curse. So we should keep in mind that the correction term is equal to zero for this specific case but for a general case may not be zero. So it needs to be appropriately treated. Now we focus on the right hand side of equation (4), i.e. the term $-\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$. The kinematic viscosity ν is equal to $\frac{\mu}{\rho}$ and the wall shear stress τ_w is equal to $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$; so the term $-\nu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ becomes equal to $-\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{\tau_w}{\rho}$. Using this expression in equation (4) we get

$$\frac{d}{dx}\int_0^\delta \left(u^2 - u u_\infty\right)dy = -\frac{\tau_w}{\rho} \tag{5}$$

Diving both sides of equation (5) by u_∞^2, we get $\frac{\tau_w}{\rho u_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy$ and the integral $\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy$ is nothing but the momentum thickness θ. So, $\frac{\tau_w}{\rho u_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy = \frac{d\theta}{dx}$ is the well-known momentum integral equation. If there is a pressure gradient term, then one trivial term will be added to this. But to illustrate the concept of the method, the best way is to consider the example of a flat plate. Now the question arises about how we can solve this equation. To solve this equation, we need to have an approximation of the velocity profile. We do not know $\frac{u}{u_\infty}$ as a function of $\frac{y}{\delta}$ but we can always make some approximation. The hope is that the boundary conditions are satisfied. Let us assume that the expression $\frac{u}{u_\infty}$ has some error because of the approximation, then the expression 1-$\frac{u}{u_\infty}$ will reduce that error since 1-$\frac{u}{u_\infty}$ is complementary to $\frac{u}{u_\infty}$. Not only that, but if the expression $\frac{u}{u_\infty}$ is integrated over the domain from 0 to δ, then the error in $\frac{u}{u_\infty}$ will be much arrested. Figure 3 shows a representative plot for $\frac{u}{u_\infty}$ as a function of $\frac{y}{\delta}$ where both the exact solution and the approximate solutions are shown. Clearly there is a difference between these two solutions, but the integral essentially talks about the area under the curve. The difference between the area under curve of the two solutions is shown by the shaded portion in figure 3. The difference is actually a little bit compensated because of the multiplication of the expression $\left(1-\frac{u}{u_\infty}\right)$. So it shows that despite the function $\frac{u}{u_\infty}$ being erroneous, the

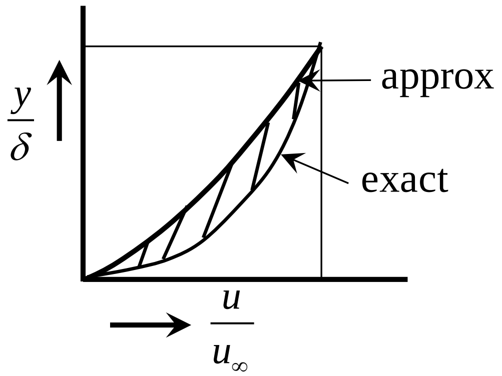

Figure 3. A representative plot of $\frac{u}{u_\infty}$ as a function of $\frac{y}{\delta}$ where both exact solution and the approximate solutions are shown.

integral of the function may be quite accurate and that is the premise of using an approximation of $\frac{u}{u_\infty}$ which satisfies some of the essential boundary conditions

depending upon the order of $\frac{u}{u_\infty}$ as a polynomial that we approximate. In the next chapter we will make some approximate choices of $\frac{u}{u_\infty}$; maybe we will work out one example. Then we will see that how we can calculate the wall shear stress and the other parameters based on that approximate choice of $\frac{u}{u_\infty}$.