Advanced Concepts In Fluid Mechanics
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Lecture - 25
Similarity Solution of Boundary Layer Equation

In the previous chapter we have discussed about the basics of boundary layer, the boundary layer theory and the boundary layer equations. In the present chapter, we will consider the boundary layer equations for flow over a flat plate and will discuss about how we can solve this equation. There is a numerical way of solving this equation and that is done after a suitable transformation which is known as the similarity transformation. First we will try to get into the basics of similarity transformation and how it originates from the physics of the problem.

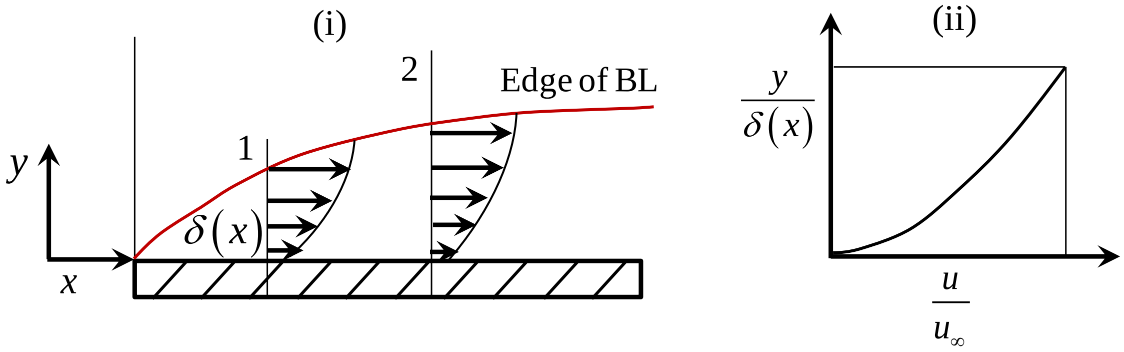

Figure 1. (i) Schematic of the flow over a flat plate where the boundary layer of thickness $\delta(x)$ grows along the *x* direction. (ii) The normalized plot of $\frac{u}{u_\infty}$ as a function of $\frac{y}{\delta(x)}$.

Now let us imagine that there is a flat plate like what is drawn in figure 1. There is a boundary layer of thickness $\delta(x)$ which grows along the *x* direction and there is also the edge of the boundary layer shown by the brown-colored solid line in figure 1. Now if we plot the velocity profile at different axial location *x*, we can see that the velocity profiles are different at different *x*. This is visible from figure 1 if one compares the velocity profiles at two sections, section 1 and section 2 respectively. The simple reason of this is the variation of the boundary layer thickness δ as if it is getting stretched as one move from section 1 to section 2. So the velocity profile is sort of getting stretched (one can imagine of a string being stretched). So the distribution of the velocity u in the *y* direction is different for different values of *x*. But if one makes a plot of $\frac{u}{u_\infty}$ as a function of $\frac{y}{\delta}$, then both the *y* axis as well as the *x* axis are normalized between 0 and 1. The entire result is within a box of dimension 1. So, interestingly, while the dimensional velocity profiles are different at different axial locations *x*, all these velocity profiles are converted into a single dimensionless velocity profile. Now question may arise about where the *x*-dependence of the velocity profile has gone which is clearly reflected earlier in the dimensional plots 1 (i). The answer is the *x*-dependence is indeed present in the non-dimensional plot 1(ii) within the parameter δ which is a function of *x*. Now we define a variable η as $\eta = y\,g(x)$ where $g(x)$ scales as $\sim\frac{1}{\delta}$. If we are able to transform the partial differential equation, which describe the boundary layer, in terms of ordinary differential equation expressed with η as an independent variable, then we can say that the problem is self-similar. In that case the similarity transformation exists. With this understanding, we define $\frac{u}{u_\infty} = f(\eta)$ where $\eta = y\,g(x)$. Now we write the boundary layer equations assuming two-dimensional incompressible and steady flow

$$\left.\begin{array}{ll} \text{Continuity equation:} & \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ x\text{-momentum equation:} & u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=\nu\dfrac{\partial^2 u}{\partial y^2} \end{array}\right\} \tag{1}$$

Now we will write various terms of the *x*-momentum equations based on this assumption that $\frac{u}{u_\infty}$ is a single valued function of η where η is essentially a variable that scales with $\frac{y}{\delta}$. Substituting $\eta = y\,g(x)$, the term u of the *x*-momentum becomes equal to $u_\infty f$. Similarly, the term $\frac{\partial u}{\partial x}$ becomes $u_\infty \frac{df}{d\eta} y\,g'(x)$. In the similar way, the term $\frac{\partial u}{\partial y}$ becomes $u_\infty \frac{df}{d\eta} g(x)$ and the term $\frac{\partial^2 u}{\partial y^2}$ becomes $u_\infty\, g^2(x)\frac{d^2 f}{d\eta^2}$. Now the strategy will be to eliminate the velocity v from the two equations $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ and

$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$ because the velocity component v is not included in the representation of $\frac{u}{u_\infty}=f(\eta)$. So we will eliminate v. In the x-momentum equation, the component v can be written as $v=\frac{\nu\frac{\partial^2 u}{\partial y^2}-u\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\nu u_\infty\ g^2(x)\frac{d^2 f}{d\eta^2}-u_\infty{}^2\ f\frac{df}{d\eta}y\,g'(x)}{u_\infty\frac{df}{d\eta}g(x)}$.

Therefore, $v=\frac{\nu\frac{d^2 f}{d\eta^2}g^2}{g\frac{df}{d\eta}}-\frac{u_\infty\ f\ y\,g'}{g}$. Now we will differentiate this expression of v with respect to y. Then we will equate with $-\frac{\partial u}{\partial x}$ in order to eliminate v. The differentiation of v with respect to y results $\frac{\partial v}{\partial y}=\frac{\nu g^2}{\cancel{g}}\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)\cancel{g}-\frac{u_\infty g'}{g}\left[\frac{df}{d\eta}g\,y+f\right]$, so,

$\frac{\partial v}{\partial y}=\nu g^2\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)-u_\infty\,g'\,y\frac{df}{d\eta}-u_\infty\ f\frac{g'}{g}$. This is equal to $-\frac{\partial u}{\partial x}$ which can be rewritten as $-\frac{\partial u}{\partial x}=-u_\infty\frac{df}{d\eta}y\,g'$. Now we equate the expressions of $\frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial x}$. Finally we get, $\nu g^2\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)-u_\infty\frac{df}{d\eta}y\,g'-u_\infty\ f\frac{g'}{g}=-\ u_\infty\frac{df}{d\eta}y\,g'$. So the term $-u_\infty\frac{df}{d\eta}y\,g'$ gets cancelled out from both sides and we get the simplified form of the ordinary differential equation as

$$\nu g^2\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)-u_\infty\ f\frac{g'}{g}=0 \tag{2}$$

We can write equation (2) in a different form as $\frac{1}{f}\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)=\frac{u_\infty}{\nu}g^{-3}\frac{dg}{dx}$. Here the left hand side $\frac{1}{f}\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)$ is a function of η only and the right hand side $\frac{u_\infty}{\nu} g^{-3}\frac{dg}{dx}$ is a function of *x* only. Since these two are equal with each other, each must be equal to a constant, so, $\frac{1}{f}\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)=\frac{u_\infty}{\nu} g^{-3}\frac{dg}{dx}=c$ where *c* is a constant. Equating $\frac{u_\infty}{\nu} g^{-3}\frac{dg}{dx}$ with *c* we get, $g^{-3}dg=\frac{c\nu}{u_\infty}dx$. Integrating both sides we get $\frac{g^{-2}}{-2}=\frac{c\nu x}{u_\infty}+c_1$ where c_1 is an integration constant. Now we need to know about the variable g. This is a very important physical question; *g* scales with $\frac{1}{\delta}$. We need to remember that at the edge of the boundary layer where *x* is equal to zero δ is not defined since it is a singular point. So, at $x\to 0^+$, i.e. at the edge of the boundary layer, we have $\delta\to 0$ which means that $g\to\infty$. Substituting this condition in $\frac{g^{-2}}{-2}=\frac{c\nu x}{u_\infty}+c_1$, we get $c_1=0$. So this expression is simplified to the form $\frac{1}{g^2}=\frac{-2c\nu x}{u_\infty}$, so, $g=\sqrt{\frac{u_\infty}{-2c\nu x}}$. From here we need to conclude about the value of the constant *c*. The value of *c* can be anything but it has to be negative because *g* is physically scaling with ~ $\frac{1}{\delta}$. So, *g* has to be positive and to make *g* positive, *c* must be negative. So if *c* is negative we could choose a good number like $-\frac{1}{2}$ or -2 like that. Let $c=-\frac{1}{2}$ which is not a must but we can choose this and then *g* becomes equal to $\sqrt{\frac{u_\infty}{c\nu x}}$, i.e. $g=\sqrt{\frac{u_\infty}{\nu x}}$. Since g scales with $\frac{1}{\delta}$ we get $\delta\propto\sqrt{x}$, this is how the boundary layer thickness grows with *x* given that the other parameters remain the same. Our main objective is not just to find this scaling relationship but to find the velocity profile which comes from the equation $\frac{1}{f}\frac{d}{d\eta}\left(\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right)=c=-\frac{1}{2}$, i.e. $d\left[\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right]+\frac{1}{2}f\,d\eta=0$. If we integrate this, an integro-differential type of equation is obtained. To make it completely differential equation we can define $\int f\,d\eta$ as a ne variable *F*, i.e. $F=\int f\,d\eta$. In other words, it can be written as $f=\frac{dF}{d\eta}$. We need to remember that f physically is velocity and $\frac{dF}{d\eta}$ is the cross gradient of some function *F*. So, *F* is physically like the stream function. In this context we recall the definition of the stream function $u=\frac{\partial\psi}{\partial y}$ which is valid for two-dimensional incompressible flow. We will try to establish an analogy between $u=\frac{\partial\psi}{\partial y}$ and $f=\frac{dF}{d\eta}$. Here u is represented by f and ψ is represented by F. Although this is purely a mathematical derivation that we have to go through, but we should never lose the physical meaning of various parameters that we are going to introduce through this equation. With this, now, we get $F'''+\frac{1}{2}FF''=\text{constant}=k$. Now the question is about the value of constant *k*. To understand that value we have to keep in mind that *F* is nothing but the stream function physically. F'' is nothing but the first derivative of velocity and F''' is the second derivative of velocity physically. Now to get the value of *k*, the relationship between *F* and the other physical variables need to be clearly kept in mind. Now we recall the boundary layer equation which is $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$. We can apply this equation at the wall. At the wall the velocity u $(u=u_w)$ is equal to zero because of the no-slip condition. The velocity *v* is also equal to zero at the wall because of the no-penetration condition. So both the terms on the left hand side $u\frac{\partial u}{\partial x}$ and $v\frac{\partial u}{\partial y}$ are equal to zero which means that the term in the right hand side $\nu\frac{\partial^2 u}{\partial y^2}$ is also equal to zero. If $\nu\frac{\partial^2 u}{\partial y^2}$ is also equal to zero at the wall then F''' is also equal to zero at $\eta=0$ which is the wall. Now we need to think of the stream function *F* at the wall. The wall is itself a streamline because by the definition there cannot be any flow across the streamline. So, by dentition the wall is like a streamline since there cannot be any such flow until and unless there are holes in the wall. So, when we do not have holes at the wall, then the wall itself represents a ψ = constant line which is a streamline. So, ψ = constant can be chosen as anything, but for

our benefit, we choose $F = 0$ at $\eta = 0$. We could have chosen anything instead of 0 (like 1,2 etc.). Whatever we have chosen then the value of k would be dependent on this particular value at the wall. So to fix it up, it is best to set it zero so that the value at the wall does not matter and that means the constant k is equal to zero. So we are left with $F''' + \frac{1}{2}FF'' = 0$ which is the famous Blasius equation. This is a non-linear third order ordinary differential equation. So we need three boundary conditions to solve this. The first boundary condition is at $\eta = 0$, the stream function $F = 0$. The secondary boundary condition is at $\eta = 0$, $f = 0$ which means that $\frac{dF}{d\eta} = 0$. The final boundary condition is at $\eta \to \infty$, $\frac{u}{u_\infty}$ is equal to 1 which means $f = 1$ or $\frac{dF}{d\eta} = 1$. S with these three boundary conditions we can numerically solve the equation $F''' + \frac{1}{2}FF'' = 0$ using various techniques such as shooting method for example. We have a separate tutorial where it will be illustrated how to numerically solve this equation. So we are not getting into that in this chapter. So this will give us a solution for F. Now we make a plot of the velocity profile $\frac{u}{u_\infty}$ as a function of η as shown in figure 2. Here $\frac{u}{u_\infty}$ is equal to 1 when η is roughly equal to 5. This is obtained from the numerical solution of the Blasius equation.

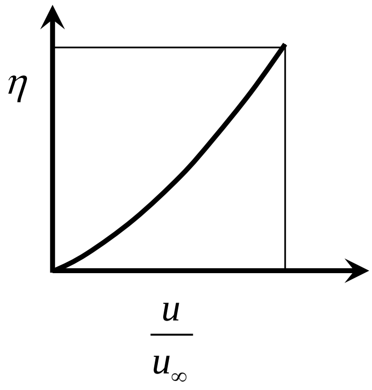

Figure 2. The variation of the velocity profile $\frac{u}{u_\infty}$ as a function of the variable η where the solution has been obtained numerically for the equation $F''' + \frac{1}{2}FF'' = 0$.

Let us write the solution again which is $\eta = y\, g(x)$ and at $\eta \approx 5$, $y = \delta$, that is the main result. It also means that $5 = \delta\, g(x)$ where the expression of $g(x)$ is given by $g(x) = \sqrt{\frac{u_\infty}{\nu\, x}}$. So, $\frac{\delta}{x} = 5\sqrt{\frac{\nu\, x}{u_\infty x^2}} = 5\ \mathrm{Re}_x^{-1/2}$ where $\mathrm{Re}_x = \frac{u_\infty\, x}{\nu}$. This is local Reynolds number. This shows the strength of the order of magnitude analysis. The order of magnitude analysis can show the scaling in just one line by equating the inertia force and the viscous force at the edge of the boundary layer, i.e. $\frac{\delta}{x}$ is of the order of $\sim \mathrm{Re}_x^{-1/2}$. The Blasius solution is only giving one additional thing which is the constant 5.

Now we will define two parameters one of which is the skin friction coefficient $C_f = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}$ which is basically the non-dimensional wall shear stress. τ_w is equal to $\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$, i.e. $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ which can be further rewritten as $\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu u_\infty \left.\frac{df}{d\eta}\frac{\partial \eta}{\partial y}\right|_{y=0} = \mu u_\infty \left.\frac{df}{d\eta}\right|_{\eta=0} g = \mu u_\infty \left. F''\right|_{\eta=0} g$ (since we have to get the solution in terms of our variables, we have expressed $\frac{\partial u}{\partial y}$ as the product $\frac{df}{d\eta}$ and $\frac{\partial \eta}{\partial y}$).

So the ratio becomes $C_f = \dfrac{\mu u_\infty F''\big|_{\eta=0} g}{\frac{1}{2}\rho u_\infty^2}$. The value of $F''\big|_{\eta=0}$ will be equal to 0.332 if we numerically calculate it. If we substitute the expression of *g*, then the skin friction coefficient becomes $C_f = 0.664\,\mathrm{Re}_x^{-1/2}$. Finally we will calculate something which is called as the drag coefficient.

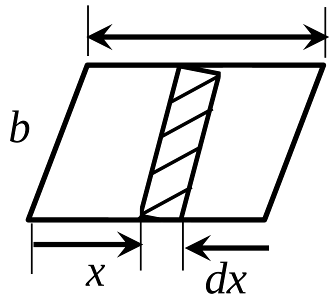

Figure 3. The figure is showing a plate which is required for the drag force calculation. Here a differential element of thickness '*dx*' has been taken into account.

To calculate the drag coefficient let us first consider the plate in which a differential element of thickness '*dx*' has to be taken into account which is taken at a distance *x*. The drag force on this differential element is $dF_D = \tau_w b\,dx$ where *b* is the width of the plate. Then the total drag force is given by $F_D = \int_0^L \tau_w b\,dx$ where *L* is the length of the plate. The drag coefficient is given by $C_D = \dfrac{F_D}{\frac{1}{2}\rho u_\infty^2 bL}$; this is a general definition of the drag coefficient which is equal to the drag force divided by a reference area. The reference area is the area which is physically meaningful to give rise to the drag force. Here the surface area of the plate is physically meaningful to give rise to the drag force. Then, we can write $C_D = \dfrac{F_D}{\frac{1}{2}\rho u_\infty^2 bL} = \dfrac{\int_0^L \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0} b\,dx}{\frac{1}{2}\rho u_\infty^2 bL}$. The term '*b*' gets cancelled out from the numerator and the denominator. Then, $C_D = \dfrac{\mu u_\infty F''\big|_0 \int_0^L \sqrt{\dfrac{u_\infty}{\nu x}}\,dx}{\frac{1}{2}\rho u_\infty^2 L} = 4F''\big|_0 \mathrm{Re}_L^{-1/2}$ where we have used the expression of g, i.e. $g(x)=\sqrt{\dfrac{u_\infty}{\nu x}}$. Here, Re_L is the Reynolds number which is based in L, i.e. $\mathrm{Re}_L = \dfrac{u_\infty L}{\nu}$. Substituting the value of $F''\big|_0$ which is equal to 0.332, the value of the drag coefficient becomes $C_D = 4\times 0.332\ \mathrm{Re}_L^{-1/2} = 1.328\ \mathrm{Re}_L^{-1/2}$. So, the constant in the skin friction coefficient C_f will always be the double of $F''\big|_0$ and the drag coefficient C_D will be four times of $F''\big|_0$. In case of the skin friction coefficient C_f , the scale is with respect to the local length scale x while in case of the drag coefficient C_D , the scale is with respect to the length scale L. We will continue from this in the next chapter.