**Advanced Concepts In Fluid Mechanics**
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**Lecture – 24**
**Introduction to Boundary Layer Theory**

In this chapter we will discuss about the Boundary Layer Theory. Boundary Layer Theory is one of the outstanding revolutionary theories that has come in the history of fluid mechanics. We need to understand the perspective of this theory before coming into a conclusion about how important or critical this theory has been. First we will discuss about the motivation of learning this theory. Starting from the design of aircraft wings to the understanding of how a cricket ball swings, how a bird flies; all these things can be well addressed if we know about the boundary layer theory and the methodology to solve the boundary layer equations. This is the vastness of the boundary layer theory. It is a subject if properly used can give rise to such a critical physical cum mathematical understanding of Aerospace Science and Engineering which cannot be really addressed by simply looking into the full form of the Navier Stokes equations. Before getting into the details of boundary layer theory we have to understand about the boundary layer and then the theory aspect comes.

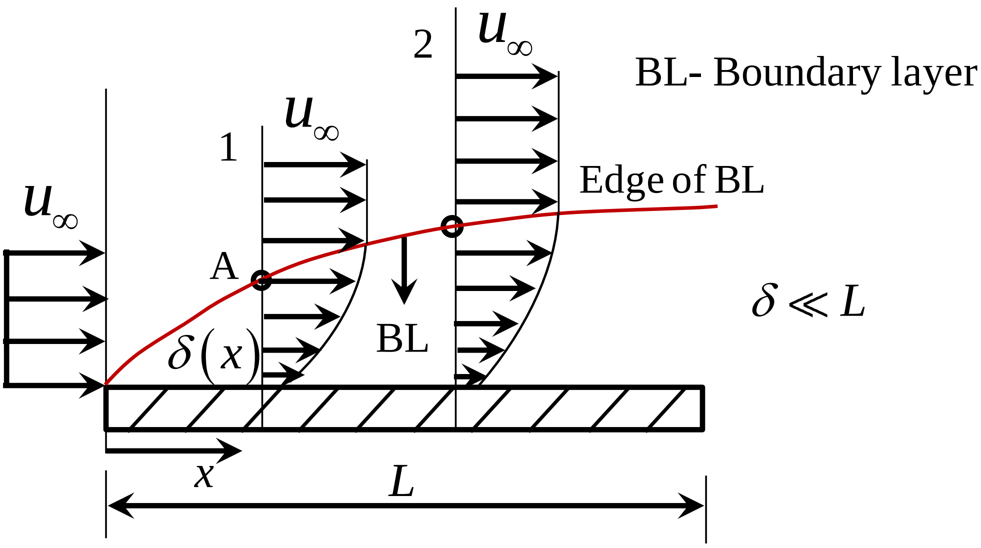


Figure 1. Schematic of the flow over a flat plate of length $L$ where the boundary layer of thickness $\delta$ grows in the axial direction.

To understand about the boundary layer let us consider that there is a flat plate which is infinitely long. We have just considered a finite length of it, i.e. length $L$ as shown in figure 1 (it can be whatever length). The fluid is coming from the fat stream with a velocity $u_\infty$ which is a uniform velocity. Next we consider about the phenomenon that

happens when the fluid interacts with the plate. Let us consider a section like section 1 what is drawn in the figure. At the wall, there is no slip between the fluid and the solid boundary. This is the point which is questionable under certain circumstances. It need not be taken as a ritual, it is just a very common situation encountered in engineering. So the velocity is equal to zero at the wall. Here the fluid responds completely to the momentum disturbance imposed by the solid boundary and is arrested to rest. If we come to a point which is a little bit away from the wall then the velocity is not equal to zero but also not equal to $u_\infty$. Now as we go further away and away, the fluid is not directly in contact with the solid boundary. But the fluid understands that there is a solid boundary. Now the question is how the fluid understands that there is a solid boundary. The answer is that there is a messenger of the momentum disturbance which is called as viscosity. Through that messenger the fluid understands that there is a momentum disturbance. In this way the velocity increases till it becomes equal to $u_\infty$ and then the velocity does not change.

Now let us go the other section (i.e. section 2 of the figure 1). If one compares the velocity profiles at the two sections, section 1 and section 2, one can notice that at the same layer the velocity is less in section 2 as compared to section 1 because more and more fluid is now in contact with the solid boundary. Then the velocity increases and finally reaches the velocity $u_\infty$. But here also it reaches the velocity $u_\infty$ at higher height in section 2 as compared to section 1. After reaching $u_\infty$ the velocity remains constant. We need to consider it from a practical engineering consideration because theoretically it reaches $u_\infty$ only at infinite distance. Practically it may reach $u_\infty$ within a finite distance and it does that then it simplifies our situation considerably. We will see that later on. For the time being we will just focus on the basic understanding of the boundary layer. The circled location $A$ shown in figure 1 is the penetration depth up to which the effect of the plate is felt. The thickness of the penetration depth is denoted by $\delta(x)$. Since the effect of the plate is felt only within the penetration depth, outside, because of the uniform velocity, the fluid does not understand where the plate is. Now we will draw the locus of the circled points of the figure (this locus is shown by the brown-colored solid line in the figure) we come up with two regions. The region within the penetration depth $\delta\left(x\right)$ is the region where the effect of viscosity is important in terms of creating a

velocity gradient. So this region is called as the boundary layer and the locus is called as the edge of the boundary layer. In short we write BL which stands for the boundary layer. We can clearly see from the figure that $\delta$ is a function of the axial co-ordinate *x*. Now the next obvious question is about the thickness of $\delta$; i.e. how small or how large this $\delta$ is. There is a possibility that this $\delta$ can be very large. $\delta$ can be very large when the fluid is highly viscous. In that case up to a large distance from the wall the effect of viscosity, i.e. the effect of the momentum disturbance of the wall will be felt. However, if the fluid is less viscous then the effect of viscosity or the momentum disturbance becomes less. The things we are saying about the thickness of $\delta$ is a qualitative representation. Quantitatively we need to identify some quantitative parameters. But the change in the thickness of $\delta$ does not have any quantitative meaning. It is just a qualitative way in which we can bring out the physics. So if the fluid is less viscous, then $\delta$ becomes very thin. Now if the thickness of $\delta$ becomes so small that its thickness at the length *L* becomes very small as compared to the length *L*, i.e. $\delta_L << L$ then we can develop a nice theory where we can solve simplified versions of the Navier Stokes equation within the boundary layer. Outside the boundary layer, we can use simple potential flow equations or inviscid irrotational flow equations which for constant density of fluid become Bernoulli equation.

So, the entire domain where ideally Navier Stokes equation should have been solved now gets reduced to a very narrow domain within which the Navier Stokes equation in a simplified form need to be solved. Outside this narrow region the Navier Stokes equation need not be referred to at all and inviscid equation can be used. In the modern era of Computational Fluid Dynamics (CFD) people may argue about the reason that why we need to enforce such restriction in the domain that the viscous flow equations will be solved only within a small part of the domain and will not be solved in the other part of the domain. People can say that they can blindly solve the full viscous flow equation (i.e. the full Navier Stokes equation) for the entire domain since we have nice CFD tool or software. Now we need to think of the era when the boundary layer theory was developed. That era was a time when the modern day high performance computing was not available. So, when modern day high performance computing was not available, in those days machine was not beating human beings; human intellect was manifested at its best. So, human intellect always tried to make an attempt to reduce the computational task by using judicious combination of physics and mathematics, and that was first attempted by the famous engineer known as Prandtl and he came up with this revolutionary theory. It is revolutionary because prior to Prandtl's era, fluid mechanics was governed primarily by the mathematicians and it was understood that there are certain classes of problems for which the exact solution of the Navier Stokes equation exists. If the exact solution does not exist then people were debating about what could be the possible solutions since there were no computational tools available. But those cases could not be addresses to come up with solutions that engineers can use for designing of devices. For example, the designing of aircrafts or designing of automobiles were not possible until this beautiful theory (boundary layer theory) appears. This theory not only reduces the computational task to a large extent but also gives a nice physical insight to the problem. It shows that irrespective of how large the domain may be, under certain cases, there is a small part of the domain in which the viscous effects are important and in the major part of the domain we can use the dynamics of inviscid flow equations. The interesting thing is that although the fluid still has a viscosity there it does not have a velocity gradient and that makes the shear stresses vanish. So this brings us to the perspective of studying the boundary layer theory. Here people may make another argument that in Prandtl's era may be CFD was not that developed and in the modern era there is no requirement of studying the boundary layer theory because we can run the CFD codes. Here we need to think that in some cases we may have a solid boundary, may not be as simple as a flat plate. When we have a solid boundary, to understand the velocity gradients at the solid boundary we need to use very fine messing or very fine grid points close to the solid boundary. But in the outer part, outside the boundary layer such fine grid points may not be required otherwise it will unnecessarily add to the computational cost. Therefore, there is always a question that how much distance close to the solid boundary we need to use the find grid and beyond what distance the fine messing is not necessary. We will know this only when we can make an assessment of the distance $\delta$ for a given physical problem. Then only we can use fine grid up to that distance and outside that we will not unnecessarily put huge computational burden by putting large number of grids. So the moral of the story is that even in the modern era of

CFD, the physical basis of the boundary layer theory remains as important as it was when it was first introduced. So with this little bit of background we will try to write the Navier Stokes equation appropriate to this physical consideration. We will first discuss about the considerations that we need to keep in mind. So our assumptions will be steady flow, the next assumption will be constant physical properties; we will also include homogeneous isotropic fluid considerations for the sake of understanding. Next is the assumption of Newtonian fluid which means that by a single parameter viscosity we can describe the constitutive behavior. If we assume the constant property then it also means that the density of the fluid is constant which implies that the derivative of density is equal to zero, .i.e. an incompressible flow. Now we will consider the two-dimensional scenario where we have the *x* and *y* co-ordinates. If we have a flat plate, then the *x* co-ordinate will be along the plate and the *y* co-ordinate will be normal to the plate as shown by the figure in the left hand side of figure 2. Now instead of flat, if it is curved, then we will not have global *x* and global *y*. In that case we will have a curve fitted *x* and *y*, we will have a co-ordinate system where *x* and *y* are relatively orthogonal but their orientations are continuously changing as we are moving along the solid boundary as shown in the right and side of figure 2. But for a flat plate, it will remain as global *x* and global *y*.

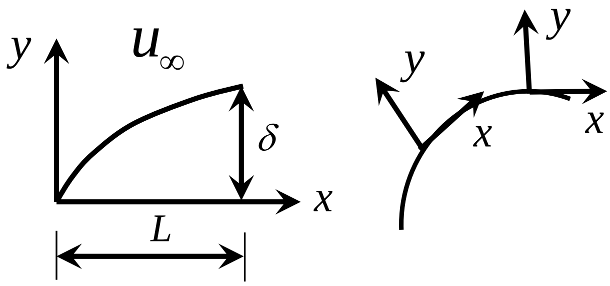


Figure 2. Co-ordinate system for the flow over a flat plate. Co-ordinate system where we have curve fitted *x* and *y* which are relatively orthogonal to each other.

First we will write the conservation of mass or the incompressibility condition in this case. The incompressibility condition is given by $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Physically we will have a boundary layer of thickness $\delta$ like what is shown in left hand side of figure 2. We have a free stream velocity $u_\infty$ and the length of the plate is *L*. We will make the order of magnitude analysis of this equation. Whatever we have studied about the order of magnitude analysis in the earlier chapters we will apply the same here. The scale of the velocity $u$ is given by ~ $u_\infty$ while the axial length scale is of the order of ~ $L$. So, $\frac{\partial u}{\partial x}$ is of the order of ~$\frac{u_\infty}{L}$. Let us assume the velocity scale for the transverse velocity $v$ is $v_\infty$ which occurs at the edge of the boundary layer. So we have a nice transition form a mathematical based theory to an engineering based theory. When we say mathematical based theory, the thickness $\delta$ will be technically equal to infinity. But in case of an engineering based theory, if the velocity reaches 99% of $u_\infty$, for all practical purposes we can assume that the condition of free stream has been reached. So we will have a finite boundary layer thickness instead of a mathematically imposed infinite boundary layer thickness. So, the term $\frac{\partial v}{\partial y}$ becomes of the order of ~ $\frac{v_\infty}{\delta}$ where $v_\infty$ is the velocity $v$ at the far stream. The two terms $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ must cancel each other to make the summation equal to zero, so, $\frac{u_\infty}{L}$ and $\frac{v_\infty}{\delta}$ must be of the same order. So, $\frac{u_\infty}{L}$ is of the order of ~ $\frac{v_\infty}{\delta}$ from which we get $v_\infty \sim u_\infty \frac{\delta}{L}$. One important thing we can understand that $v_\infty$ becomes much less than $u_\infty$, i.e. $v_\infty \ll u_\infty$ if $\delta$ is much less than $L$, i.e. $\delta \ll L$. But $v_\infty$ is never equal to zero no matter how much less the thickness $\delta$ is as compared to the length $L$. This is a big difference between this and the fully developed flow. In case of a fully developed flow, we have the velocity $v$ equal to zero. But in the present case $v$ may be very small as compared to the velocity $u$ but it is not identically zero. Some students have a misunderstanding that $v$ is equal to zero because when we draw the velocity profile in the boundary layer, we only draw the $x$ component of the velocity profile, we do not draw the $y$ component of the velocity profile. But the $y$ component is very much present there. If $\delta$ is comparable with the length $L$, then $v_\infty$ and $u_\infty$ may be comparable. But in boundary layer theory we are looking for only those conditions in which $\delta$ is much less than $L$. So the boundary layer will exist for all viscous flow problems. It may be very small or may be as large as infinity but the boundary layer theory is a theory which captures only those problems where the boundary layer thickness is much less than the length $L$. In the other part there may be boundary layer but the boundary layer theory will not work. So in the present case, $v_\infty \ll u_\infty$ since $\delta$ is much less than $L$. Now we will consider the $x$-momentum and the $y$-momentum equations. First we will write the $x$ and $y$ momentum equations and then we will perform the order of magnitude analysis. The two momentum equations are given below

$x$-momentum: $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

and $y$-momentum: $$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

where the assumptions of steady flow, constant properties and Newtonian fluid have been taken into account. At this stage we know about how to write the Navier Stokes equation. Let us assume that there is no body force. Now let us find the order of magnitudes of the different terms of these two momentum equations. Since $u$ and $x$ are of the order of $u_\infty$ and $L$ respectively, the term $u\frac{\partial u}{\partial x}$ in equation (1) becomes of the order of $\sim\frac{{u_\infty}^2}{L}$. The second term $v\frac{\partial u}{\partial y}$ becomes of the order of $\sim\frac{v_\infty u_\infty}{\delta}$. But from the incompressibility condition we know that $v_\infty \sim u_\infty\frac{\delta}{L}$. Substituting this, we get that the term $v\frac{\partial u}{\partial y}$ to become of the order of $\sim\frac{{u_\infty}^2}{L}$. This is something which is not intuitive. So the term $u\frac{\partial u}{\partial x}$ and the term $v\frac{\partial u}{\partial y}$ are of the same order. Natural intuition tells us that since the velocity $v$ is much less than $u$ because $\delta$ is much less than $L$, the term $v\frac{\partial u}{\partial y}$ is much less than $u\frac{\partial u}{\partial x}$ which is not true. This is not true because the gradient $\frac{\partial u}{\partial y}$ is much sharper than the gradient $\frac{\partial u}{\partial x}$. So the tem $v\frac{\partial u}{\partial y}$ is exactly as important as the term $u\frac{\partial u}{\partial x}$. If we cannot ignore the term $u\frac{\partial u}{\partial x}$ we cannot ignore the term $v\frac{\partial u}{\partial y}$ also. Now we focus on the right side of the $x$-momentum equation (1). In between the two terms $\nu\frac{\partial^2 u}{\partial x^2}$

and $\nu \frac{\partial^2 u}{\partial y^2}$, $\nu \frac{\partial^2 u}{\partial x^2}$ is of the order of $\frac{\nu u_\infty}{L^2}$ and $\nu \frac{\partial^2 u}{\partial y^2}$ is of the order of $\frac{\nu u_\infty}{\delta^2}$. Since $\delta$ is very small as compared to the length *L*, the term $\nu \frac{\partial^2 u}{\partial x^2}$ becomes very small as compared to the term $\nu \frac{\partial^2 u}{\partial y^2}$ and one can neglect the term $\nu \frac{\partial^2 u}{\partial x^2}$. At this moment we are not commenting about the pressure gradient $\frac{\partial p}{\partial x}$ (i.e. about its order of magnitude) which will be discussed later. Now we will find the order of magnitudes of different terms of the *y*-momentum equation (i.e. equation (2)) in a similar way as it is done for the *x*-momentum equation. The first term $u \frac{\partial v}{\partial x}$ is of the order of $\sim \frac{u_\infty v_\infty}{L}$. Using the continuity equation (or the incompressibility condition), the second term will also be of the same order, i.e. $v \frac{\partial v}{\partial y} \sim \frac{u_\infty v_\infty}{L}$. Now the term on the right side $\nu \frac{\partial^2 v}{\partial x^2}$ will be of the order of $\sim \nu \frac{v_\infty}{L^2}$ while the term $\nu \frac{\partial^2 v}{\partial y^2}$ will be of the order of $\sim \nu \frac{v_\infty}{\delta^2}$. Since $\delta \ll L$, the term $\nu \frac{\partial^2 v}{\partial x^2}$ can be neglected as compared to the term $\nu \frac{\partial^2 v}{\partial y^2}$. All the considerations here are based on the fact that $\delta$ is much less than *L*, the entire understanding is based on that. So let us compare the inertial terms (i.e. the terms on the left hand side of the equations) of the *x*-momentum and the *y*-momentum equations. For *x*-momentum, the inertial terms are of the order of $\sim \frac{u_\infty{}^2}{L}$ while for the *y*-momentum, the inertial terms are of the order of $\sim \frac{u_\infty v_\infty}{L}$. So, $\frac{\text{inertial term in the } y\text{-momentum}}{\text{inertial term in the } x\text{-momentum}} \sim \frac{\frac{u_\infty v_\infty}{L}}{\frac{u_\infty{}^2}{L}} \sim \frac{v_\infty}{u_\infty} \sim \frac{\delta}{L}$ which is much less than 1 if $\delta$ is much less than *L*. Similarly, if we compare the viscous terms of the two momentum equations, we get $\frac{\text{viscous term in the } y\text{-momentum}}{\text{viscous term in the } x\text{-momentum}} \sim \frac{\nu \frac{v_\infty}{\delta^2}}{\frac{\nu u_\infty}{\delta^2}} \sim \frac{v_\infty}{u_\infty} \sim \frac{\delta}{L}$. So the same conclusion about the viscous terms can be drawn if $\delta$ is much less than *L*. So, if the inertial terms of the *y*-momentum equation are at least one order of magnitude less than the inertial terms of the $x$-momentum equation and the viscous terms of the $y$-momentum equation are at least one order of magnitude less than the viscous terms of the $x$-momentum equation, the remaining term $\frac{\partial p}{\partial y}$ (of $y$-momentum) will be at least tone order of magnitude less than the term $\frac{\partial p}{\partial x}$ (of $x$-momentum). So, the important conclusion is $\frac{\partial p}{\partial y} \ll \frac{\partial p}{\partial x}$. This is the conclusion after performing the order of magnitude analysis in the $y$-momentum equation. In that case we can write $p$ as a function of $x$ only. So, the term $\frac{\partial p}{\partial x}$ can be approximated as $\frac{dp}{dx}$. So this leads us to the boundary layer equations. From the Navier Stokes equation it leads to a simplified Navier Stokes equation within the boundary layer which is called as the boundary layer equations as given below

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned}\right\} \tag{3}$$

So now we have the boundary layer equations and the question arises about when these equations will be valid. To understand this, we take the example of flow over a flat plate. Of course, we have taken into consideration the fact that $\delta$ is much less than $L$. But there can be a physical problem where some external condition is imposed which does not understand about the thickness of $\delta$. So we need to express this in terms of externally controllable physical parameters. Considering the flow over a flat plate as an example, since there is no pressure gradient in the boundary layer, we can write $\frac{dp}{dx}$ to be equal to $\frac{dp_\infty}{dx}$. Thus the pressure gradient along the $x$ direction becomes equal to that imposed from the outside of the boundary layer. In the outside of the boundary layer, the beauty is that we can use the Bernoulli's equation if the density of the fluid $\rho$ is constant. The reason is that the uniform flow is irrotational outside the boundary layer and it is inviscid also. Inviscid flow means the irrotational flow will remain irrotational forever. So we can use the Bernoulli's equation, i.e. $p_\infty + \frac{1}{2}\rho u_\infty^2 = C$ where $C$ is an absolute global constant.

Now if we differentiate it with respect to *x*, we get $\frac{dp_\infty}{dx} + \rho u_\infty \frac{du_\infty}{dx} = 0$. Since the far stream velocity $u_\infty$ does not change with *x*, $\frac{du_\infty}{dx} = 0$ and we get $\frac{dp_\infty}{dx} = 0$. Now

substituting $\frac{dp_\infty}{dx} = 0$ in the *x*-momentum equation we get $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$. Here, the terms in the left hand side are of the order of $\sim \frac{u_\infty^2}{L}$ while the term in the right hand side is of the order of $\sim \frac{\nu u_\infty}{\delta^2}$. These two terms $\frac{u_\infty^2}{L}$ and $\frac{\nu u_\infty}{\delta^2}$ are of the same order, i.e. $\frac{u_\infty^2}{L} \sim \frac{\nu u_\infty}{\delta^2}$ from which we get $\frac{\delta}{L} \sim \left(\frac{u_\infty L}{\nu}\right)^{-1/2}$. Now $\frac{u_\infty L}{\nu}$ is the definition of the Reynolds number, i.e. $\mathrm{Re}_L = \frac{u_\infty L}{\nu}$, so, $\frac{\delta}{L} \sim \mathrm{Re}_L{}^{-1/2}$. Now in the expression of $\frac{u_\infty L}{\nu}$ terms are externally controllable parameters. We are able to say that $\delta$ is much less than *L* only when the Reynolds number $\mathrm{Re}_L$ is large. So the boundary layer theory is applicable to large Reynolds number problem. Now the question arises about how large the Reynolds number can be. The value of the Reynolds number $\mathrm{Re}_L$ has to be such that $\delta$ is at least one order of magnitude lower than *L*; the ratio of $\frac{\delta}{L}$ must be at least 0.1 or less. From there we can find a corresponding Reynolds number. For that Reynolds number and beyond that Reynolds number we can use the boundary layer theory. So $\delta$ much less than *L* will boil down to very large Reynolds number $\mathrm{Re}_L$.

Now the question arises about when the boundary layer theory is applicable. The answer is that the boundary layer theory is applicable when $\frac{\delta}{L}$ is much less than unity (i.e. $\frac{\delta}{L} << 1$) which is equivalent to large Reynolds number $(\mathrm{Re}_L)$. There can be other cases which may not be encountered in the case of flow over a flat plate like the pressure gradient $\frac{dp}{dx}$ can play a very critical role. For example, if the pressure gradient $\frac{dp}{dx}$ is such that there is an adverse pressure gradient, it means that the pressure is increasing along the *x* direction. If the pressure increases along *x*, then there will be a force which is opposite to the motion of the fluid. The fluid wants to move along the positive $x$ direction but the pressure gradient, which is called as the adverse pressure gradient, is trying to oppose the movement. Along with that there is also the viscous force which also tries to slow the fluid down. So the acceleration or the inertia of the fluid may not be sufficient enough to overcome these two forces and if it is insufficient, the fluid instead of moving forward may start to move backward along the negative $x$ direction. This is called as boundary layer separation. This boundary layer separation is possible only when there is adverse pressure gradient. If the pressure gradient is favorable, then the pressure gradient will drive the flow along the positive $x$ direction and boundary layer separation may not be possible. But if there is an adverse pressure gradient there is a chance of boundary layer separation and if it occurs then there is no more monotonic growth of $\delta$ as a function of $x$ and then we can say that the boundary layer theory does not work. So the boundary layer theory is applicable when (i) $\frac{\delta}{L}$ is much less than 1 and (ii) there is no boundary layer separation. This is not very commonly discussed but this is very important. For flow over a flat plate boundary layer separation does not matter because there is no question of an adverse pressure gradient, the pressure gradient is zero; so there is no question of boundary layer separation.

So, in the present chapter, we have learnt about the boundary layer theory, the boundary layer equations, the reason of its importance and the assumptions that are behind this boundary layer theory. We will take it forward from this in the next chapter.