Advanced Concepts In Fluid Mechanics
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Lecture – 23
Statistical Treatment of Turbulence and Near – Wall Velocity Profiles

In the previous chapter, we have discussed about some of the physical features of the turbulent flow. Now the question arises about how we can represent these physical features or take these physical features into account through a mathematical approach. It is very difficult to have a rigorous mathematical approach for turbulent flow because we have a wide range of length scales and time scales (that is one of the physical features). So, if we want to computationally solve a problem, we may need to resolve a domain which should be able to capture from the smallest scale (i.e. the molecular scale) up to the large eddy scale; all the relevant spatial and temporal scales. Although this can be said easily, it is not so easy to implement because it requires huge computational effort. There are established strategies known as direct numerical simulation strategies through which the Navier-Stokes equation can directly be used to solve turbulent flow problems. We need to remember that the Navier-Stokes equations are very much valid for turbulent flow provided the other conditions are satisfied. It is a question of implementing the numerical scheme that makes it so challenging. Certain statistically based models have come up which hold the capability of solving the turbulent flow problems in an approximate sense which serves most of the engineering purposes.

Now we will discuss about the statistical treatment of turbulent flows. Let us say that there is a powerful probe or a device which is plotting one component of velocity. We need to remember that turbulent flow, because of its random fluctuations over space and time is always three-dimensional and unsteady. There is nothing called as steady turbulence. We will discuss later that what we mean as steady turbulence in terms of the physical understanding but the terminology is not very appropriate to consider.

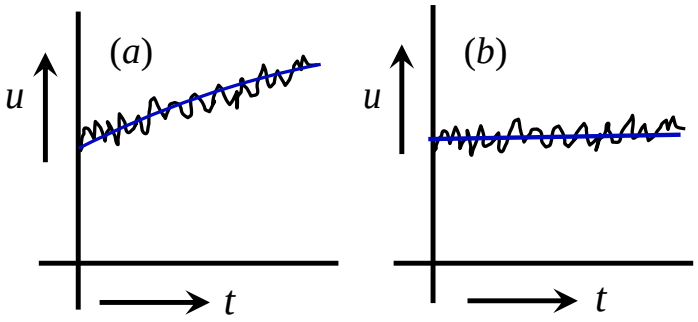

Figure 1. The variation of velocity u with time t for two different scenarios (a) and (b) respectively. The blue-colored solid lines represent the corresponding statistical averages.

Now we make two plots for two different scenarios. In the first case, the plot is shown by the black-colored curve in figure 1(a) while in the other case the plot is shown by the black-colored curve in figure 1(b). These representations are very difficult to capture vividly through a model because we can see that there are very small oscillations of u over time. Similarly, if we draw a variation of u over space similar things can be inferred. Now we want to make a statistical average of this variation. We define the statistical average of velocity as $\bar{u} = \underset{T\to\infty}{Lim}\frac{1}{T}\int_0^T u\,dt$. This is essentially an integration of u over time t from 0 to a time period of T which tends to infinity. Similar to this variable u, we can use any other variable like v, w, p etc. Now the definition of statistical average needs to be clearly understood; it does not literally mean that T will be equal to infinity. It means that this limit of T is significantly larger as compared to the small fluctuation times over which it is fluctuating. If that be the case, then we can smoothen out these data in terms of an average $\bar{u}$. These average values of the data are shown by the blue-colored solid lines in figures 1(a) and 1(b). We should remember that T is much larger than the small fluctuation time scale but it also has to be smaller than the total time otherwise we cannot capture this variation of the average value over time. If we make an average which does not vary with time although u varies with time, then sometimes it is erroneously called as steady turbulence. This type of variation is shown by the blue-colored solid line in figure 1(b). Instead of this time averaging, averaging can also be done similarly over the space. For time averaging, we have u at a given location and we are integrating with respect to time. In space averaging, we integrate u with respect to space at a given time. We are not putting any bias; this is just the standard way of calculating the average. No matter whether it is time average or the space average, the concept remains similar.

Then if we have a function *f*, we can resolve it in terms of average $\overline{f}$ and fluctuation f', i.e. $f = \overline{f} + f'$. So, as an example, the difference between *u* and $\overline{u}$ is given by u'. This is called as Reynold's decomposition. Now if we take average on both sides of $f = \overline{f} + f'$, the average of f becomes equal to $\overline{f}$ and the average of $\overline{f}$ is also $\overline{f}$. From this, we get the average of the fluctuation f' to be equal to zero. But the interesting thing is that although the average of f' is zero and average of another function g' is zero, the average of the product $f'g'$ is not equal to zero. The reason that the two terms are equal to zero when averaged individually but when their products are averaged there is no guarantee that it will be equal to zero. With this little bit of background on the statistics of turbulent flow, we will define few other important terminologies, for example, root mean square (rms) of velocity *u*. Root mean square of *u* is defined as $\sqrt{(u-\overline{u})^2}$ which is equal to $\sqrt{u'^2}$ and this is not zero definitely. Now we can formulate the Navier-Stokes equation by decomposing it into the mean and the fluctuating component and this is called as the Reynolds averaging of the Navier-Stokes equation. The advantage with the Reynolds averaging is that we do not require any more to capture the smallest length scales and the smallest time scales. So we can consider a length scale and a time scale which are substantially elevated as compared to the finest ones which are physically there. So computationally the problem becomes more tractable. Now we will write the Navier-Stokes equation in the conservative form which is given by $\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + b_i$ where b_i is the body force term. The important thing that needs to be mentioned is that we will not depend so much on the body force. We now decompose the variables into mean and fluctuating components to get $\frac{\partial}{\partial t}\left[\rho(\overline{u}_i + u_i')\right] + \frac{\partial}{\partial x_j}\left[\rho(\overline{u}_i + u_i')(\overline{u}_j + u_j')\right] = -\frac{\partial\left(\overline{p}_i + p_i'\right)}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial(\overline{u}_i + u_i')}{\partial x_j}\right) + b_i$

velocity u_i into two components as $u_i = \overline{u}_i + u_i'$. In the next step we will do an averaging of this equation; we get a '¯' symbol over the terms to indicate the averaging and we get $\overline{\frac{\partial}{\partial t}\left[\rho(\overline{u}_i + u_i')\right]} + \overline{\frac{\partial}{\partial x_j}\left[\rho(\overline{u}_i + u_i')(\overline{u}_j + u_j')\right]} = -\overline{\frac{\partial\left(\overline{p}_i + p_i'\right)}{\partial x_i}} + \overline{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial(\overline{u}_i + u_i')}{\partial x_j}\right)} + b_i$. One

important factor missed in this calculation is that the density ρ should also be ideally decomposed into mean and fluctuating components because density may also be fluctuating in the turbulent flow. But we assume a constant density flow and that's why we did not decompose it into two components. But one needs to remember that we cannot do that for all cases. For our interests, density is not fluctuating; it may have a larger time scale over which it varies. It is still ok but it is not fluctuating like the other parameters *u*, *v* and *p*. So, if we take the average, the average of fluctuating component will be equal to zero. We should remember that the time scale over which we are finding out the average is different from the time derivative $\frac{\partial}{\partial t}$. So this time derivative and the time averaging do not conflict with each other. The term $\overline{\frac{\partial}{\partial x_j}\left[\rho\left(\bar{u}_i+u_i'\right)\left(\bar{u}_j+u_j'\right)\right]}$ is a little bit of special one and we will discuss it later on. After time averaging, $\overline{\frac{\partial}{\partial t}\left[\rho\left(\bar{u}_i+u_i'\right)\right]}$ will be equal to $\frac{\partial}{\partial t}\left(\rho\bar{u}_i\right)$, the term $\overline{\frac{\partial\left(\bar{p}_i+p_i'\right)}{\partial x_i}}$ will be equal to $\frac{\partial\bar{p}_i}{\partial x_i}$ and the term $\overline{\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\left(\bar{u}_i+u_i'\right)}{\partial x_j}\right)}$ will be equal to $\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\right)$. Now we will work out the remaining term $\overline{\frac{\partial}{\partial x_j}\left[\rho\left(\bar{u}_i+u_i'\right)\left(\bar{u}_j+u_j'\right)\right]}$. If we expand the product $\left(\bar{u}_i+u_i'\right)\left(\bar{u}_j+u_j'\right)$ we get four terms as $\left(\bar{u}_i+u_i'\right)\left(\bar{u}_j+u_j'\right)=\bar{u}_i\bar{u}_j+\bar{u}_i u_j'+\bar{u}_j\bar{u}_i'+u_i'u_j'$. The averaging of the term $\bar{u}_i\bar{u}_j$ will result $\bar{u}_i\bar{u}_j$. Now, when we make an average of the term $\bar{u}_i u_j'$, $\bar{u}_i$ being constant with respect to the average will come out of averaging and it will become the integral of u_j' over the time period which will be equal to zero. In the similar way, when we make an average of the term $\bar{u}_j\bar{u}_i'$, $\bar{u}_j$ being constant with respect to the average will come out of averaging and it will become the integral of $\bar{u}_i'$ over the time period which will be equal to zero. But the average of the product of two fluctuations $u_i'u_j'$ will not be equal to zero. So, $\overline{\frac{\partial}{\partial x_j}\left[\rho\left(\bar{u}_i+u_i'\right)\left(\bar{u}_j+u_j'\right)\right]}$ will become simplified to the form

$\frac{\partial}{\partial x_j}\left[\rho\,\bar{u}_i\,\bar{u}_j\right]+\frac{\partial}{\partial x_j}\left[\overline{\rho u_i' u_j'}\right]$. So we can, in principle, write a Reynolds averaged Navier-Stokes equation (or in the short form called as RANS) which will be in the following form $\frac{\partial}{\partial t}\left(\rho\bar{u}_i\right)+\frac{\partial}{\partial x_j}\left(\rho\bar{u}_i\bar{u}_j\right)=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\right)+b_i-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$. This equation looks like the Navier-Stokes equation in an averaged sense except for the additional term $-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$. Because of dimensional equivalence of $-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$ with the term $\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\right)$, $-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$ indicates the Reynolds stress or the turbulent stress. $\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\right)$ can be treated as τ_{ij} and $-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$ can be rewritten as $\frac{\partial}{\partial x_j}\left[-\rho\overline{u_i' u_j'}\right]$, then $-\rho\overline{u_i' u_j'}$ represents the expression of the Reynolds stress or the turbulent stress. This is purely done from the analogy of the dimension of this additional term $-\frac{\partial}{\partial x_j}\left[\rho\overline{u_i' u_j'}\right]$ with the dimension of the viscous term $\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\right)$. So, while trying to get rid of problem of solving a turbulent flow we have come up with the situation where we have observed some additional unknowns in the form of second order tensor. To overcome this problem one possibility is that we write the expression $-\rho\overline{u_i' u_j'}$ in the form $-\rho\overline{u_i' u_j'}=\mu_T\frac{\partial\bar{u}_i}{\partial x}$ where μ_T is a fuzzy parameter called as turbulent viscosity. It is not a constant but a parameter which depends on the turbulent fluctuations which can be mathematically modeled. So, μ_T is not a constant but it needs to be close with the aid of other turbulence parameters. This is called a closer problem in turbulence modeling. It tells that the Navier-Stokes equations after Reynolds averaging does not remain closed because of the introduction of the turbulent stress and we need a closer model for μ_T (which is called as a turbulent viscosity) to close the problem. We will talk about certain models which achieves this but before that we will try to discuss some special cases of the statistical approach.

The first one is the homogeneous turbulence. Homogeneous turbulence is a case where the turbulence statistics (for example, average, RANS etc.) are not functions of position. We have already talked about the homogeneous term but the homogeneous turbulence only talks about the homogeneity in the statistical representation, not the homogeneity in the parameters because the parameters are all distributed over position. So, homogeneous turbulence means that the turbulence properties are not dependent on position which means that they are invariant to the translation of coordinate axis since we can change the position by translating the coordinate axis. So this is translational invariance.

Similarly, isotropic turbulence talks about rotational invariance. Not only that, by definition, it is invariant with respect to translation as well as reflection (all sorts of transformations) which preserve the length. So, the statistics are invariant with respect to that. Therefore, isotropic turbulence by definition must be homogeneous because there is a translational invariance associated with it. These types of turbulences are important because we can conclude something about the averaging technique. In real experiments, we neither purposefully do the time averaging nor we purposefully do the space averaging; but we do ensemble averaging.

So question arises about the ensemble averaging. We usually do a large number of repeated experiments under identical conditions. It is very important to do repeated experiments under identical conditions. So, if we repeat the experiments under identical conditions, then we call the averaging as the ensemble averaging. If it is a homogeneous turbulence, then we can say that the ensemble average is same as the space average because it does not vary with space. So repeating experiments at different spatial locations is as good as repeating the experiments under identical conditions. Similarly, if we have stationary turbulence, then the ensemble average becomes equal to time average because things do not vary with time (the turbulence statistics does not vary with time). Now, if we have homogeneous as well as stationary turbulence it will mean that the time average is equal to the space average which is further equal to the ensemble average. This is known as the Ergodic hypothesis. So, if this Ergodic hypothesis is satisfied, the averaging we use could be time average, space average, ensemble average whatever. Now we will discuss a little bit about the closer model to show the procedure to close these system of equations. We will now talk about one classical model which is called as Prandtl's mixing length model.

Clearly from the dimensional arguments and by drawing analogy from the molecular theory, Prandtl proposed that the kinematic viscosity can be thought of as a product of a length scale over which there is turbulent mixing multiplied by the fluctuation component of the velocity, i.e. $\nu_t = l_m u'$. Here ν_t is the kinematic viscosity, l_m is the length scale over which there is turbulent mixing and u' is the fluctuating component of velocity. ν_t is nothing but the turbulent viscosity μ_T divided by the density ρ. Now the problem is that we neither know l_m nor we know u'. We can have a different way of looking into it; we can take the form $|u'| = l_m \left|\frac{\partial \bar{u}}{\partial y}\right|$. Then the expression of ν_t will be $\nu_t = l_m^{\,2} \left|\frac{\partial \bar{u}}{\partial y}\right|$. Here, y is the direction in which the mixing takes place. Let us say we have some eddies which are interacting with other set of eddies. There is a fluctuating velocity in the *y* direction because of which the eddies are interacting. If one part is moving slower then it will try to slower the other part down. So the mixing length can be modeled as $\nu_t = l_m^{\,2} \left|\frac{\partial \bar{u}}{\partial y}\right|$ which tells that the kinematic viscosity can be expressed in terms of the mixing length. If this mixing length can be somehow estimated then we can close the analysis. There are various other closer models. For example, we can calculate the kinetic energy (*k*) based on the fluctuating component of the velocity; we can calculate the dissipation rate of kinetic energy which is denoted by ε. We can have two additional equations on k and ε; then we can relate μ_T with k and μ_T which is called as k-ε model. So, there are various models which exist. The present course is not a course on turbulence modeling but we are trying to provide the philosophy behind this. The philosophy is that we have to somehow close the analysis; we have to somehow get an expression of μ_T in terms of the other parameters. So we have only one parameter l_m using which we can close the system. There is one very important issue that no matter whether the flow is laminar or turbulent, the flow very close to the wall is always laminar. So wall-bounded turbulent flows are very important and we will now briefly discuss about it.

Let us consider that we have a wall, the y direction is in the upward direction. In a region very close to the wall, we can write $\tau_w = \mu \frac{\bar{u}}{y}$. This is called as the near-wall

behavior in the viscous sub-layer region. In some books it is called as the laminar sub-layer but it is not correct. The reason is that in the viscous layer, the flow is predominantly laminar but there are some small turbulent fluctuations in that layer (although that does not affect the wall shear stress $\tau_w = \mu \frac{\bar{u}}{y}$). From dimensional analogy we can write the wall shear stress τ_w as $\rho u_\tau^{\,2}$; then $\frac{u}{u_\tau} = \frac{\rho u_\tau y}{\mu} = \frac{y}{\frac{\mu}{\rho u_\tau}}$. Now we define $\frac{u}{u_\tau}$ as u^+ and $\frac{y}{\frac{\mu}{\rho u_\tau}}$ as y^+. Very adjacent to the wall we will have $u^+ = y^+$. Of course, after we go little bit of away from the wall, we will have u^+ as a different function of y^+, i.e. $u^+ = f\left(y^+\right)$. But when we go to the outside little bit far away from the wall, then the turbulent stresses will come into picture. So, $u^+ = f\left(y^+\right)$ is like the inner behavior and the outer layer behavior is given by $\frac{\bar{u} - u_\infty}{u_\tau} = f\left(\eta\right)$ which is called as the velocity defect law. We get all these from dimensional argument. So, $\frac{\bar{u} - u_\infty}{u_\tau} = f\left(\eta\right)$ is called as velocity defect law and we will show later that how this law relates to the condition of $u^+ = f\left(y^+\right)$. So, $u^+ = f\left(y^+\right)$ is an inner behavior and $\frac{\bar{u} - u_\infty}{u_\tau} = f\left(\eta\right)$ is an outer behavior. The deviation of $\bar{u}$ from the free stream velocity u_∞ is because of the turbulent stress which is somehow normalized with respect to u_τ. In this way, the inner layer effect propagates to the outer layer. $\frac{\bar{u} - u_\infty}{u_\tau}$ is a function of η where η is a different dimensionless scale $\eta = \frac{y}{\delta}$ where δ is the length scale of the outer layer (it has nothing to do with the inner layer). We have a continuity of $\frac{du}{dy}$ in the inner layer and the outer layer, i.e. $\left.\frac{d\bar{u}}{dy}\right|_{\text{inner}} = \left.\frac{d\bar{u}}{dy}\right|_{\text{outer}}$. We can write the left hand side as $\left.\frac{d\bar{u}}{dy}\right|_{\text{inner}} = \frac{d\bar{u}}{dy^+}\frac{dy^+}{dy} = \frac{d\bar{u}}{dy^+}\frac{\rho u_\tau}{\mu}$. Similarly, we can write the right hand side as

$\left.\frac{d\bar{u}}{dy}\right|_{\text{outer}} = \frac{d\bar{u}}{d\eta}\frac{d\eta}{dy} = \frac{d\bar{u}}{d\eta}\frac{1}{\delta}$. Now we play a simple trick; we multiply both sides by *y* to get $\frac{d\bar{u}}{dy^+}\frac{\rho u_\tau}{\mu}y = \frac{d\bar{u}}{d\eta}\frac{y}{\delta}$. Now the term $\frac{\rho u_\tau}{\mu}y$ is nothing but equal to y^+ and the term $\frac{y}{\delta}$ is nothing but η, so, $\frac{d\bar{u}}{dy^+}y^+ = \frac{d\bar{u}}{d\eta}\eta$. This is about the overlapping region. Now, $\frac{d\bar{u}}{dy^+}y^+$ is a function of y^+ only; it does not understand η. Similarly, $\frac{d\bar{u}}{d\eta}\eta$ is a function of η only; it does not understand y^+. Since they are equal to each other; they must be equal to a constant, so, $\frac{d\bar{u}}{dy^+}y^+ = \frac{d\bar{u}}{d\eta}\eta = \text{constant}$. We rewrite this as $\frac{du^+}{dy^+}y^+ = \frac{du^+}{d\eta}\eta = \text{constant}$, i.e. we switch from $\bar{u}$ to u^+ for our understanding since we have used y^+. Since, in the both sides there was $\bar{u}$, changing it to u^+ will not make any difference in the expressions. Now if we integrate $\frac{du^+}{dy^+}y^+ = \frac{du^+}{d\eta}\eta = \text{constant}$, we will get $u^+ = \frac{1}{\kappa}\ln y^+ + B$ which is called as the logarithmic law or the log law. Remarkably this $\frac{1}{\kappa}$ is a universal constant and κ is called as Von Karmann constant. For all experiments, the value of $\frac{1}{\kappa}$ is found to be around 2.5 which is remarkable but the experiments have revealed that. The constant *B* is around 5. These κ and *B* are the universal constants. Now, if we sum this up, we will get a variation like what is drawn in figure 2. It shows the variation of u^+ as a function of y^+ in a logarithmic graph. Up to $y^+ = 5$, there will be one kind of variation, up to this $y^+ = 5$, we have the viscous layer for which we have $u^+ = y^+$. Beyond this value of y^+, u^+ will be a different function of y^+ (not $u^+ = y^+$). Up to the limit $y^+ = 30$, we will still have this kind of behavior. In the outer layer we have the logarithmic law which reads as $u^+ = \frac{1}{\kappa}\ln y^+ + B$. So, if we plot u^+ as a function of $\ln y^+$, then it will be a straight line. If we plot the real physical data, then those data will smoothly converge in this regions and then beyond some y^+, those physical data will have a significant deviation from the behavior described by figure 2. So the region

up to $y^+ = 5$, i.e. $0 \le y^+ \le 5$ is known as the viscous layer and $5 < y^+ \le 30$ is called as the buffer layer. This figure also shows the inner layer as well as the outer layer from which one can see that there is an overlap between the inner layer and the outer layer. So the overlapping region is the region where the logarithmic law holds. Red lines represent the fitted law and green lines represent the experimental data and we can see that how remarkably the near-wall behavior varies under certain conditions.

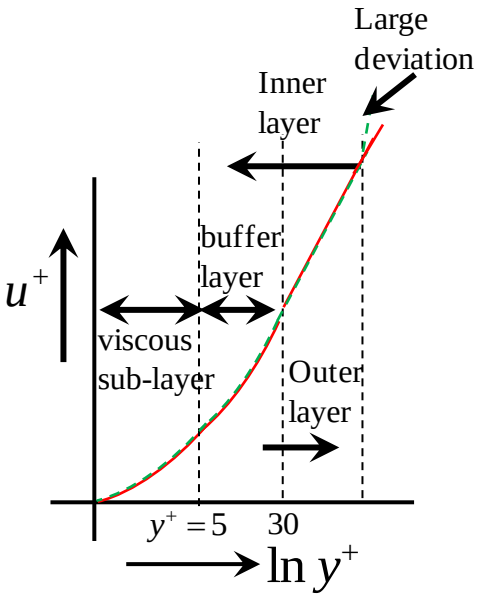

Figure 2. The variation of u^+ as a function of $\ln y^+$ where different regions are shown.

Overall, we have discussed the fundamental aspects of turbulent flow in the present chapter and the previous chapter. There are many practical engineering applications where the concept of turbulent flow and laminar flow and their combination are used to design a body for minimizing the drag forces. We will have a separate tutorial where there will be practical considerations for designing of aerospace vehicles or designing of engineering object by minimizing the drag force based on the concept of laminar flow and the turbulent flow that we have learned in this fluid mechanics course.