Advanced Concepts In Fluid Mechanics

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Lecture - 21
Confined Oscillatory Flows

In the previous chapter we have discussed about how an oscillation in a solid boundary can influence the flow in the surroundings. However those surroundings were considered to be unconfined. In the present chapter we will consider an oscillatory flow in a confinement which can be thought of the first primitive model to understand how an oscillatory flow of blood takes place in blood vessels. So here we will consider the oscillatory flow in a confinement.

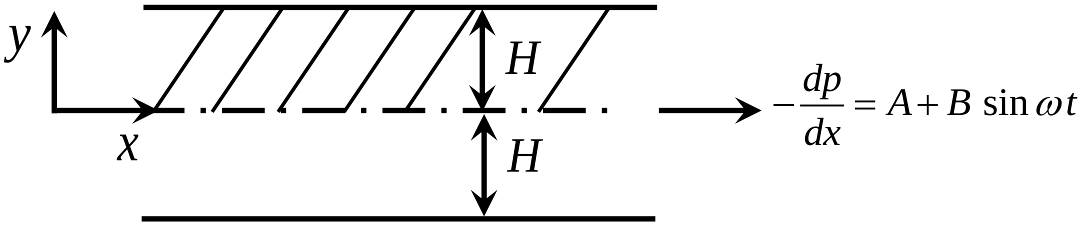

Figure 1. Schematic of an oscillatory flow in a confinement where we have a pressure gradient $-\frac{dp}{dx} = A + B\sin\omega t$ acting along the *x* direction.

Let us consider a channel of height $2H$ as shown in figure 1. For simplicity we have considered a rigid channel. We have to keep in mind that the real blood vessel is flexible rather than being rigid although there are lots of controversial issues and unknown paradoxes about the flexibility of blood vessel and its universal nature. For simplicity we will consider it as a rigid channel. We have a pressure gradient $-\frac{dp}{dx} = A + B\sin\omega t$ acting along the *x* direction. This is a reasonable form, of course we can add sin and cosine terms to make it more practical but mathematically this gives a physically relevant signature to solve the problem. At the walls we have no-slip boundary condition, the half-height of the channel is H. The pressure gradient $\frac{dp}{dx}$ is not a function of *x* and therefore, it is translational invariant problem along *x*. It is a function of time but not a function of *x*. The governing equation for this problem is given in the following

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

Here the only difference in the present problem with the previous problem is that here we have a pressure gradient which is acting in terms of the governing equation. In case of an unconfined flow, until and unless we apply a special pressure gradient, by the natural consequence of the flow there is no pressure gradient. But when there is a confined flow, there has to be a pressure gradient. Otherwise we cannot drive the flow until and unless we have special mechanisms like application of an electric field or other external fields to drive the flow.

Now the present problem has a steady part and an unsteady part. So we will decompose the problem into a steady problem and an unsteady problem. In the form of $-\frac{dp}{dx} = A + B\sin\omega t$, A corresponds to the pressure gradient for a steady problem. The steady part of the solution is governed by

Steady part: $$0 = A + \mu \frac{\partial^2 u}{\partial y^2} \tag{2}$$

The velocity u is a function of the y co-ordinate only, i.e. $u = u_{ss}(y)$ and therefore, $\frac{\partial^2 u}{\partial y^2}$ can be replaced by $\frac{d^2 u}{dy^2}$. So, the steady part of the velocity is denoted by the variable u_{ss}. Then the governing equation becomes $0 = A + \mu \frac{d^2 u}{dy^2}$. This steady problem we have already solved in previous chapters where a pressure driven flow in a parallel plate channel has been considered. So here we will not discuss it again. We will only mention about the boundary conditions. Because of the symmetry of the problem only half of the domain can be chosen for solving. At the channel centerline, i.e. at $y = 0$, we have $\frac{du}{dy} = 0$ which is known as the symmetry condition. At $y = H$, the velocity u is equal to zero because of no-slip boundary condition. Now we will focus on the unsteady part of the solution. The governing equation for the unsteady part is given below

Unsteady part: $$\rho \frac{\partial \hat{u}}{\partial t} = B\sin\omega t + \mu \frac{\partial^2 \hat{u}}{\partial y^2} \tag{3}$$

where the unsteady part of the velocity is denoted by the variable $\hat{u}$. The final solution of the problem is the summation of the steady part u_{ss} and the unsteady part $\hat{u}$, i.e. $u = u_{ss} + \hat{u}$. In the governing equation of the steady part, instead of $0 = A + \mu \dfrac{d^2u}{dy^2}$ we can also write it as $\rho \dfrac{\partial u_{ss}}{\partial t} = A + \mu \dfrac{d^2u_{ss}}{dy^2}$ where the time derivative term $\rho \dfrac{\partial u_{ss}}{\partial t}$ for the steady part is equal to zero. Now, as usual, we will first make a scaling analysis to the problem before attempting for a solution for the unsteady part. To do this, let us first write the governing equation, i.e. $\rho \dfrac{\partial \hat{u}}{\partial t} = B \sin \omega t + \mu \dfrac{\partial^2 \hat{u}}{\partial y^2}$. This problem is not like that a plate is oscillating. So we do not have a natural velocity scale. The velocity scale will depend on the pressure gradient but it is not explicit. Since we do not know the velocity scale, we simply write the scale of velocity $\hat{u}$ as $\sim u_s$ while the scale of time t is obviously $\sim t_s \sim \dfrac{1}{\omega}$. So, the scale of the left hand side of equation $\rho \dfrac{\partial \hat{u}}{\partial t} = B \sin \omega t + \mu \dfrac{\partial^2 \hat{u}}{\partial y^2}$ becomes $\sim \rho u_s \omega$. The term $B \sin \omega t$ is of the order of $\sim B$ because $\sin \omega t$ is of the order of unity. The scale of the term $\mu \dfrac{\partial^2 \hat{u}}{\partial y^2}$ is $\sim \mu \dfrac{u_s}{H^2}$. Since we are writing orders of magnitude whether it is H or $2H$ it does not matter. Using these scales we can non-dimensionalize the variables as $\bar{t} = \dfrac{t}{t_s}$, $\bar{u} = \dfrac{\hat{u}}{u_s}$ (with a question mark about what the velocity scale u_s is) and $\bar{y} = \dfrac{y}{H}$. Now the dimensionless form of the governing equation $\rho \dfrac{\partial \hat{u}}{\partial t} = B \sin \omega t + \mu \dfrac{\partial^2 \hat{u}}{\partial y^2}$ becomes $\rho u_s \omega \dfrac{\partial \bar{u}}{\partial \bar{t}} = B \sin \bar{t} + \dfrac{\mu u_s}{H^2} \dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}$. Now in this problem two terms will always be important out of the three terms $\rho u_s \omega \dfrac{\partial \bar{u}}{\partial \bar{t}}$, $B \sin \bar{t}$ and $\dfrac{\mu u_s}{H^2} \dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}$. If we are interested about an oscillatory flow, then the term $B \sin \bar{t}$ has to be important because that is what we are aiming for to create an oscillation in the flow. If this term $B \sin \bar{t}$ is not important we will never get an oscillation in the flow. So $B \sin \bar{t}$ becomes important according to our requirement. There is something which is intrinsically important which is the unsteady represented by the term $\rho u_s \,\omega \dfrac{\partial \overline{u}}{\partial \overline{t}}$. When we are creating an oscillation we must have unsteadiness in the flow. Now we do not know about the remaining term $\dfrac{\mu u_s}{H^2}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2}$, (i.e. the viscous term) whether it is

important or not. That depends on the kinematic viscosity and the other parameters and that is the interesting physics of this problem.

Now we will divide both sides by B we get $\dfrac{\rho u_s \,\omega}{B}\dfrac{\partial \overline{u}}{\partial \overline{t}} = \sin \overline{t} + \dfrac{\mu u_s}{BH^2}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2}$. Here $\sin \overline{t}$ term is of the order of 1 (i.e. ~O(1)). In order to become equally important, the term $\dfrac{\rho u_s \,\omega}{B}\dfrac{\partial \overline{u}}{\partial \overline{t}}$ should be of the order of 1 in which the non-dimensional term $\dfrac{\partial \overline{u}}{\partial \overline{t}}$ is already of the order of 1. So the term $\dfrac{\rho u_s \,\omega}{B}$ must be of the order of 1 such that the product of $\dfrac{\rho u_s \,\omega}{B}$ and $\dfrac{\partial \overline{u}}{\partial \overline{t}}$ becomes equally important with the term $\sin \overline{t}$ otherwise it cannot compete with the term $\sin \overline{t}$. If $\dfrac{\rho u_s \,\omega}{B}\dfrac{\partial \overline{u}}{\partial \overline{t}}$ is of the order of 0.1 and $\sin \overline{t}$ is of the order of 1, then obviously $\dfrac{\rho u_s \,\omega}{B}\dfrac{\partial \overline{u}}{\partial \overline{t}}$ cannot compete. So with this, we get the two terms $\dfrac{\rho u_s \,\omega}{B}\dfrac{\partial \overline{u}}{\partial \overline{t}}$ and $\sin \overline{t}$ equally important. From $\dfrac{\rho u_s \,\omega}{B} \sim \mathrm{O}(1)$ we can find out $u_s \sim \dfrac{B}{\rho\,\omega}$. So this gives a natural velocity scale of the problem. Although there is no velocity which is directly imposed from the oscillatory component of the pressure gradient we can get a velocity scale natural to the problem. Once we substitute this scale of u_s in the remaining term $\dfrac{\mu u_s}{BH^2}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2}$ the term gets modified as $\dfrac{\mu u_s}{BH^2}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2} = \dfrac{\mu}{H^2 \rho\,\omega}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2}$ where B gets cancelled out from the numerator and the denominator. Now, $\dfrac{\mu}{\rho} = \nu$ is known as the kinematic viscosity, so it becomes $\dfrac{\nu}{H^2\,\omega}\dfrac{\partial^2 \overline{u}}{\partial \overline{y}^2}$. Now we need to think of the term $\dfrac{\nu}{\omega}$ and its significance. If we recall from the Stokes second problem (which was discussed in the previous chapter), $\frac{\nu}{\omega}$ is equal to the square of the penetration depth δ, i.e. $\frac{\nu}{\omega}=\delta^2$ if the plate was simply oscillated by a linear shear. Let us write $\frac{\nu}{\omega H^2}=\frac{\delta^2}{H^2}=\frac{1}{\lambda^2}$ where $\lambda=\frac{H}{\delta}$ with δ being the penetration depth corresponding to a simple oscillatory flow in an unconfined environment. The value of the parameter λ will now determine whether

this term $\frac{\mu u_s}{B H^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$ is important or not. So, we have an external control by which we can say whether the viscous term is important or not. If the value of λ is smaller than 1, then the reciprocal of λ (i.e. $\frac{1}{\lambda}$) is large and then the viscous term can be very important. Here λ is small only when δ is large and δ is large when ω is small. So we notice that whether the viscous term will be important or not, it also depends on the oscillatory frequency. So, overall it depends on three important parameters which is like a design problem in engineering. So to make the viscous term important or not important we need to look into the three parameters ω, ν and H. By making a combination of these three terms we come up with the non-dimensional parameter λ which tells us whether the viscous term is important or not. Combining all these aforesaid considerations, we write the final form of the governing equation

$$\frac{\partial \bar{u}}{\partial \bar{t}}=\sin \bar{t}+\frac{1}{\lambda^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{4}$$

Just like the Stokes 2nd problem, here also we will have an oscillatory solution. We assume $\bar{u}=\operatorname{Im}\left[e^{i\bar{t}} f\left(\bar{y}\right)\right]$, here it is imaginary because of the presence of the $\sin\bar{t}$ term. Instead of $\sin\bar{t}$ if $\cos\bar{t}$ term is present we need to take the real part of the function $e^{i\bar{t}}$, i.e. $\operatorname{Re}\left[e^{i\bar{t}}\right]$. If we take the time derivative of $\bar{u}$, then we get $\frac{\partial \bar{u}}{\partial \bar{t}}=\operatorname{Im}\left[i\, e^{i\bar{t}} f\left(\bar{y}\right)\right]$. The term $\sin\bar{t}$ can be written as $\sin\bar{t}=\operatorname{Im}\left[e^{i\bar{t}}\right]$ while the term $\frac{1}{\lambda^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$ can be written as $\frac{1}{\lambda^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}=\operatorname{Im}\left[\frac{1}{\lambda^2} f'' e^{i\bar{t}}\right]$. Substituting these expressions in equation (4) we get

$$f'' - \lambda^2 i f + \lambda^2 = 0 \tag{5}$$

To make it little bit more simple (just algebraically simple), we define $\overline{f} = f + i$, so, $i f = i \overline{f} - i^2 = i \overline{f} + 1$ since i^2 is equal to -1. Some rearrangement makes it $-i f + 1 = i \overline{f}$. Also f'' is equal to $\overline{f}''$. Substituting theses expressions in equation (5) we get $\overline{f}'' - \lambda^2 i \overline{f} = 0$. We can write the solution of this equation either in terms of exponential functions or in terms of hyperbolic functions. For a change let us write the general

solution of this in terms of hyperbolic functions as $\overline{f} = c_1 \cosh\left(\lambda\sqrt{i}\,\overline{y}\right) + c_2 \sinh\left(\lambda\sqrt{i}\,\overline{y}\right)$.

We could have written it as $\overline{f} = c_1 e^{\lambda\sqrt{i}\,\overline{y}} + c_2 e^{-\lambda\sqrt{i}\,\overline{y}}$; representation in terms of the exponential form is as good as the representation in terms of the hyperbolic terms (they are just the different combinations). In order to obtain the constants c_1 and c_2, we need to apply the boundary conditions. The first boundary condition is the symmetry condition at the channel centerline, i.e. at $\overline{y} = 0$, $\dfrac{\partial \overline{u}}{\partial \overline{y}} = 0$ which is further simplified to $\dfrac{d\overline{f}}{d\overline{y}} = 0$.

From the expression $\overline{f} = c_1 \cosh\left(\lambda\sqrt{i}\,\overline{y}\right) + c_2 \sinh\left(\lambda\sqrt{i}\,\overline{y}\right)$ we can say that $\dfrac{d\overline{f}}{d\overline{y}}$ is equal to $\lambda\sqrt{i}\left\{c_1 \sinh\left(\lambda\sqrt{i}\,\overline{y}\right) + c_2 \cosh\left(\lambda\sqrt{i}\,\overline{y}\right)\right\}$, at $\overline{y} = 0$ $c_1 \sinh\left(\lambda\sqrt{i}\,\overline{y}\right)$ is obviously equal to zero. So, in order to make $\dfrac{d\overline{f}}{d\overline{y}}$ equal to zero the other integration constant c_2 must be equal to zero. Then the function $\overline{f}$ is simplified to the form $\overline{f} = c_1 \cosh\left(\lambda\sqrt{i}\,\overline{y}\right)$. Now we apply the other boundary condition, i.e. at $\overline{y} = 1$ (or $y = H$), $\overline{u}$ is equal to zero in which it consists of both the steady part $\left(u_{ss}\right)$ as well as the unsteady part $\left(\hat{u}\right)$. Since the steady part has already been taken equal to zero, the unsteady part is also equal to zero which means $f = 0$. Using the relation $\overline{f} = f + i$, we get $\overline{f}$ being equal to i at $\overline{y} = 1$. So we get, $c_1 \cosh\left(\lambda\sqrt{i}\right) = i$ from which we get the expression of c_1 as $c_1 = \dfrac{i}{\cosh\left(\lambda\sqrt{i}\right)}$.

Hence, we have got the solution as $\overline{u} = \mathrm{Im}\left[e^{i\overline{t}}\,\dfrac{i\cosh\left(\lambda\sqrt{i}\,\overline{y}\right)}{\cosh\left(\lambda\sqrt{i}\right)}\right]$. From this expression of

$\bar{u}$ one can determine the unsteady component $\hat{u}$ and the final solution of velocity will be the summation of the steady part $\left(u_{ss}\right)$ and the unsteady part $\left(\hat{u}\right)$, i.e. $u = u_{ss} + \hat{u}$. So, only the 'cosh' term is present in the unsteady part. So, clearly the time dependence will have a very important role to play depending on the parameter λ which combines three factors ω, ν and H. Now it is suggested for the students to make a plot of the profile $\bar{u} = \mathrm{Im}\left[e^{i\bar{t}} \frac{i\cosh\left(\lambda\sqrt{i}\,\bar{y}\right)}{\cosh\left(\lambda\sqrt{i}\right)} \right]$ using some software like MATLAB where we can assume this particular form of solution and use inbuilt complex operators with which we can extract the real part and the imaginary part separately. These are inbuilt as mathematical function. We can construct the function and then take the imaginary part and then can add the steady part with it and plot the velocity distribution in the channel. To do this we need to know about the relevant parameters. In this context the utility of solving the problem in a dimensionless environment comes into picture where all the relevant parameters come through a single dimensionless term λ. One of the relevant parameters is the frequency ω, another is the kinematic viscosity ν while the remaining one is the channel height H. These three parameters are combined together in the parameter λ. So by choosing different values of λ i.e. by just a single parameter (which contains interrelationships of different parameters) we can get very interesting types of velocity distributions in the channel. Overall, in this chapter we have discussed about the role of confinement in an oscillatory flow and we have studied to a reasonable extent that how the pressure gradient can affect the flow for both steady and unsteady scenarios.