Advanced Concepts In Fluid Mechanics

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Lecture - 20
Exact Solution of Navier Stokes Equation for Unsteady Flow

In the previous few chapters, we have discussed about some exact solutions of Navier Stokes equation for steady flow. In reality many flows are unsteady flows. In the present chapter we will focus on obtaining the exact solutions for Navier Stokes equations for two special unsteady flow problems which are very classical problems. The first one is known as the Stokes 1st problem while the second one is known as the Stokes 2nd problem.

Stokes first problem:

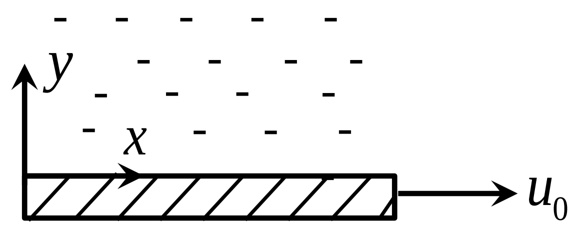

Figure 1. Schematic of the Stokes 1st problem where the plate is pulled axially with a velocity u_0 .

Let us consider that we have a flat plate as shown in figure 1. The fluid is near the plate which is standstill as the plate is not moving. Suddenly we drag the plate towards the right (or the left whatever) with the velocity u_0 and this plate tends to drag the fluid along with it. Our point of interest here is to examine that how the velocity in this fluid varies as a function of the position and time. All those traditional considerations like Newtonian fluid, Stokesian fluid are valid here also. When we say about position, it is ideally a three-dimensional flow. But if the width of the plate is infinitely long then it is still a two-dimensional problem. Now if we think of the co-ordinate system, here it is the rectangular Cartesian co-ordinate system with *x* and *y* co-ordinates. The plate is pulled along the *x* direction. Here the predominant effect is the variation of the velocity along the *y* direction. Now question may arise about when there can be the variation of the velocity along the *x* direction. The answer that when u_0 becomes a function of the axial co-ordinate *x*, then this velocity can vary in the *x* direction. Since the pulling velocity u_0 is not a function of *x*, there is no variation of this velocity in the *x* direction. So, because of the uniform pulling along the *x* direction, this problem is having a translational invariance with respect to *x*. One should not get confused between the translational

invariance and the fully developed flow. Here the velocity u is not a function of *x* $u \neq u(x)$ because of translational invariance which comes from the boundary condition. But the fully developed flow comes from a very important fundamental premise, i.e. the flow is confined in a closed passage (like channel flow or pipe flow). But the present problem is an open flow problem and hence, one cannot use the concept of the fully developed flow here. So u is not a function of *x* here; it is a function of the *y* co-ordinate and the time *t*, i.e. $u(y,t)$ (or $u(t,y)$) . If we assume two-dimensional incompressible flow, we can write the continuity equation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

Out of the two terms in the continuity equation (1), the term $\frac{\partial u}{\partial x}$ is equal to zero because of translational invariance which means that the remaining term $\frac{\partial v}{\partial y}$ is equal to zero. Now $\frac{\partial v}{\partial y}=0$ implies that the velocity component v is not a function of the *y* co-ordinate. Also the velocity v is equal to zero at *y* = 0 because of the no-penetration boundary condition. This indicates that the velocity v is equal to zero for all *y*.

In order to obtain the velocity distribution let us write the governing equation; i.e. the *x* component of the Navier Stokes equation

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

In the momentum equation (2), the terms $\rho u\frac{\partial u}{\partial x}$ and $\mu\frac{\partial^2 u}{\partial x^2}$ are equal to zero because the velocity u is not a function of *x*. The term $\rho v\frac{\partial u}{\partial y}$ is equal to zero because the velocity v is equal to zero for all *y*. Since the flow is unbounded in nature and there is infinite fluid. So everywhere there is atmospheric pressure which means that there is no pressure gradient acting in the axial direction; so the term $\frac{\partial p}{\partial x}$ is equal to zero. Using these considerations, the simplified x momentum equation can be written as

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \qquad (3)$$

This equation (3) can be written in an alternative form as $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ where $\nu = \frac{\mu}{\rho}$. Now we need to discuss about the parameter ν which is called as the kinematic viscosity of the fluid which is a very important property. The fluid viscosity μ (or dynamic viscosity) is the ability to create momentum disturbance. So when we have a plate moving, by virtue of the fluid property viscosity this message of movement of the plate is propagated to the outer fluid. But the outer fluid because of its inertia tries to retain its earlier momentum which is the condition of rest in this example. Because mass is the measure of the inertia and density ρ is related to mass, the parameter $\nu = \frac{\mu}{\rho}$ indicates the ability of the fluid to create a disturbance in the momentum relative to its ability to sustain its momentum and that is the physical meaning of the kinematic viscosity. So kinematic viscosity is a very important parameter, it is not just like the fluid viscosity. Fluid viscosity tries to diffuse momentum but there is something which tries to sustain its momentum and their relative importance is a matter of concern. So instead of just considering it as a fluidic property, it is necessary to consider its physical significance.

Now, looking into the simplified momentum equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ one can understand that this equation has certain scales involved. For the scale of the velocity u , its maximum value u_0 can be chosen. But for the time there is no such scale, time t can vary starting from zero to time tending towards infinity. So we can take time as an independent variable. Let us say that we are interested with a time scale of t which is denoted by t_s , then the scale of the term $\frac{\partial u}{\partial t}$ becomes of the order of $\frac{\partial u}{\partial t} \sim \frac{u_0}{t_s}$ (here the '~' symbol corresponds to the order of magnitude of the respective term). Here t_s can vary from zero to infinity. We are interested to observe about what is happening within the time scale t_s. Now, the scale of the term on the right hand side is given be $\nu \frac{\partial^2 u}{\partial y^2} \sim \nu \frac{u_0}{\delta^2}$ where δ is the length scale which is yet to be known. Physically δ signifies the thickness of the fluid which responds to the movement of the plate at a time $t = t_s$. Beyond this δ, the outer fluid does not understand that there is a plate which is moving. So, there the velocity remains zero. Only within this δ layer, the velocity is changing. In the scale of $\nu \frac{\partial^2 u}{\partial y^2}$ the term δ^2 appears in the denominator because of the presence of y derivative twice while in the numerator there is the presence of the velocity scale u_0, so, it becomes $\sim \nu \frac{u_0}{\delta^2}$. Since the two terms $\frac{\partial u}{\partial t}$ and $\nu \frac{\partial^2 u}{\partial y^2}$ are equal, they must be of the same order because there is no other term to cancel these two terms in the momentum equation. So if equate the orders of these two terms, we can write $\delta \sim \sqrt{\nu \, t_s}$. So at a time $t = t_s$, we would get a feel about the order of the magnitude of the thickness of the penetration layer δ. Now let us plot the variation of the velocity as a function of the position and time which is qualitatively shown in figure 2.

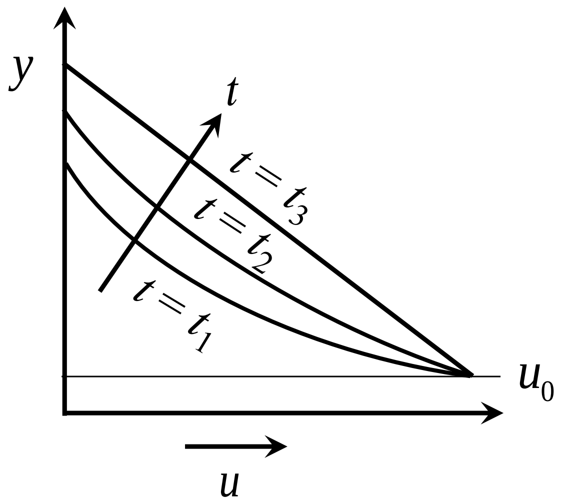

Figure 2. The variation of the velocity u as a function of the transverse co-ordinate y evaluated at different times t_1, t_2 and t_3 respectively.

In figure 2 we make a plot of the velocity variation where in the axial direction there is the velocity u while in the transverse direction there is the y co-ordinate. The velocity at the boundary is equal to u_0 as shown in the figure. This boundary condition remains in the figure. Now the variation of the velocity u is plotted for a time $t = t_1$. Then we increase the time from $t = t_1$ to $t = t_2$. In this way, we increase the time up to $t = t_3$ (the direction of increasing time *t* is also shown in the figure). Physically this figure tells that as we allow more and more time, more thickness of the fluid is responding to the movement of the plate. So the velocity plots are all scattered. If we want to collapse all these data in the form of a single variable, we need to understand the similarities of these data. The similarity is that the expression $\frac{u}{u_0}$ is a function of a single quantity η, i.e. $\frac{u}{u_0} = f(\eta)$ where η scales with $\frac{y}{\delta}$, i.e. $\eta \sim \frac{y}{\delta}$. So, our expectation is the representation of $\frac{u}{u_0}$ as a function of a single variable η and this expectation will be true only when we will be able to convert the partial differential equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ into an ordinary differential equation. If we cannot convert or we are not in a position to convert this, then we can say that similarity does not exist. Since δ is a functions of *t*, i.e. $\delta(t)$; then the expression $\frac{u}{u_0}$ also becomes function of *t*. But the expression $\frac{u}{u_0}$ is a function of the variable η which itself depends on both *y* and *t*. Let us assume $\eta = y\, g(t)$ where $g(t)$ scales with the reciprocal of the thickness $\delta(t)$. Now let us consider the equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$. Using the newly assigned variable η, the term $\frac{\partial u}{\partial t}$ can be written as $\frac{\partial u}{\partial t} = u_0 \frac{df}{d\eta} \frac{\partial \eta}{\partial t} = u_0 \frac{df}{d\eta} y \frac{dg}{dt}$. Then to determine the term $\frac{\partial^2 u}{\partial y^2}$ let us first express $\frac{\partial u}{\partial y}$ in terms of the variable η which is $\frac{\partial u}{\partial y} = u_0 \frac{df}{d\eta} \frac{\partial \eta}{\partial y} = u_0 \frac{df}{d\eta} g$. So, $\frac{\partial^2 u}{\partial y^2}$ can be written as $\frac{\partial^2 u}{\partial y^2} = u_0 \frac{d^2 f}{d\eta^2} g^2$. Since the two terms $\frac{\partial u}{\partial t}$ and $\nu \frac{\partial^2 u}{\partial y^2}$ are equal we can write $u_0 \frac{df}{d\eta} y \frac{dg}{dt} = \nu u_0 \frac{d^2 f}{d\eta^2} g^2$. The term u_0 gets canceled out from the both sides, we substitute y by $\frac{\eta}{g}$ and finally we get the following expression of the ordinary differential equation

$$\frac{\frac{d^2 f}{d\eta^2}}{\eta \frac{df}{d\eta}} = \frac{\frac{dg}{dt}}{\nu g^3} \tag{4}$$

So now we have been successful in separating the variables. The left hand side $\frac{d^2 f}{d\eta^2} \Big/ \eta \frac{df}{d\eta}$ is a function of η only and the right hand side $\frac{\frac{dg}{dt}}{\nu g^3}$ is a function of t only. So these two terms are equal only when they are equal to a constant. Let us assume c be the constant. So we can write $\frac{\frac{d^2 f}{d\eta^2}}{\eta \frac{df}{d\eta}} = \frac{\frac{dg}{dt}}{\nu g^3} = c$. Let us take the part $\frac{\frac{dg}{dt}}{\nu g^3} = c$ which can be rewritten as $g^{-3} dg = c \nu \, dt$. Now integrating this with respect to t we get $\frac{g^{-2}}{-2} = c \nu t + c_1$ where c_1 is the integration constant. Now we recall the parameter g which scales with ~ $\frac{1}{\delta}$ where δ is the penetration depth which determines about how much of the fluid understands the effect of the movement of the plate at a given time t. At time $t \to 0$, δ will be tending towards zero because the plate movement has just started. Then g will tend to infinity, i.e. $g \to \infty$. Applying the condition $t \to 0$, $g \to \infty$, the integration constant c_1 becomes equal to zero which means that $g = \frac{1}{\sqrt{-2c\nu t}}$. So, out of a rigorous mathematical exercise the simple thing that g scales with $\frac{1}{\sqrt{\nu t}}$ is recovered. Since g scales with $\frac{1}{\delta}$, δ scales with $\sqrt{\nu t}$. This highlights the strength of the order of magnitude scaling analysis that whatever could be recovered from very involved elaborate mathematics the same thing could be recovered from pure intuitive physical arguments. Not let us consider the possible values of c which can have a value of anything. But here it has to negative because g has to be real; so c must be negative. As an example we choose $c = -2$ which is not a must, we can choose any other value.

As we have learnt in the school level in ratio proportion problem that if $\frac{a}{b}$ is equal to $\frac{c}{d}$, we can choose each to be equal to a constant k. The value of k does not matter as long as the condition $\frac{a}{b}=\frac{c}{d}$ is satisfied. In the present case, the value of c does not matter as long as g remains physically realistic. If we choose $c=-2$, then 2 comes out of the square root and we get $g=\frac{1}{2\sqrt{\nu t}}$. Using the value of c and equating this with $\frac{d^2 f}{d\eta^2}\Big/\eta\frac{df}{d\eta}$ we can write

$$\frac{\frac{d^2 f}{d\eta^2}}{\eta\frac{df}{d\eta}}=-2 \tag{5}$$

Equation (5) is a very simple ordinary differential equation to solve. Let $\frac{df}{d\eta}=h$, so we have $\frac{dh}{d\eta}+2\eta h=0$. This can be rewritten as $\frac{dh}{h}=-2\eta\, d\eta$. Integrating both sides we get $\ln h=-\eta^2+\ln k$ from which we get $h=k\,e^{-\eta^2}$. Using the relation $\frac{df}{d\eta}=h$ we get the final form of the function f, i.e. $f=k\int_0^{\eta} e^{-\eta^2}d\eta+k_1$ where k and k_1 are integration constants. These two constants can be obtained by using two boundary conditions. The first boundary condition is obvious, i.e. at $\eta=0$ (which means $y=0$) we have $u=u_0$, so the function $\frac{u}{u_0}$ becomes equal to 1, i.e. $f=1$. Using this boundary condition we get $k_1=1$. Now we apply the second boundary condition. At $\eta\to\infty$, the function f becomes equal to zero because the fluid at infinity does not understand the effect of the solid boundary; then $0=k\int_0^{\infty} e^{-\eta^2}d\eta+1$. Now we need to evaluate the integral $\int_0^{\infty} e^{-\eta^2}d\eta$. Let $\eta^2=z$, so, $\eta=z^{1/2}$ and $d\eta=\frac{1}{2}z^{\frac{1}{2}-1}dz$. Substituting this in the integral we get $\int_0^{\infty} e^{-\eta^2}d\eta=\frac{1}{2}\int_0^{\infty} e^{-z}z^{\left(\frac{1}{2}-1\right)}dz$. The integral $\int_0^{\infty} e^{-z}z^{\left(\frac{1}{2}-1\right)}dz$ is a standard integral the result of which is given by $\Gamma\left(\frac{1}{2}\right)$ (where Γ is called as the gamma function). The value of $\Gamma\left(\frac{1}{2}\right)$ can be given by $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. So the final value of the integral $\int_0^\infty e^{-\eta^2}d\eta$ becomes equal to $\frac{\sqrt{\pi}}{2}$. So we get $k\frac{\sqrt{\pi}}{2}+1=0$ or, $k=-\frac{2}{\sqrt{\pi}}$. The final form of the function f upon the substitution of the constants k and k_1 is given by $f=1-\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}d\eta$. The integral $\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}d\eta$ by definition is known as the error function $erf(\eta)$. So, $f=1-erf(\eta)=erfc(\eta)$ where $erfc(\eta)$ is known as the complementary error function. So this is the solution to the problem. Now we make a graphical plot of the variation of the parameter $\frac{u}{u_0}$ as a function of the variable $\eta=\frac{y}{2\sqrt{\nu t}}$ as shown in figure 3. In case of the variation of u with respect to *y*, we get different characteristics at different time (as observed in figure 2), but here all data collapses into a single master curve. The maximum value of the variable η becomes equal to 2 while the maximum value of $\frac{u}{u_0}$ is

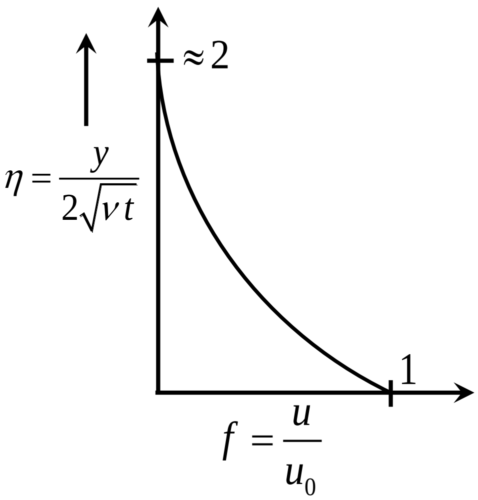

Figure 3. The Variation of $\frac{u}{u_0}$ as a function of the variable $\eta=\frac{y}{2\sqrt{\nu t}}$.

equal to 1 (i.e. $u=u_0$). We can understand that if we have more time, we have more penetration depth *y*. But beyond $\eta=2$, the outer does not understand the effect of the movement of the plate. So, this gives a nice physical insight to the problem. Although technically below $y = \infty$, any fluid should understand the effect of the plate movement, but for all practical purposes, beyond the non-dimensional thickness $\eta = 2$ the fluid does not feel the effect of the plate movement.

Stokes second problem:

Now we will consider the next problem which is known as the Stokes 2nd problem. Here the only difference with the Stokes 1st problem is that instead of the plate moving along a particular direction it is oscillating with $u = u_0 \sin \omega t$. All other considerations of the previous problem remain the same. So, the governing equation remains the same; only difference occurs in the boundary condition at $y = 0$.

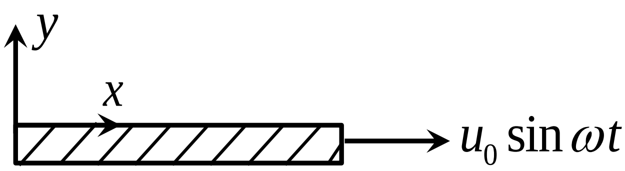

Figure 4. Schematic of the Stokes 2nd problem where the plate is oscillating with $u = u_0 \sin \omega t$.

Let us start with the governing equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$. If we do a scaling analysis of this equation, the scale of the term $\frac{\partial u}{\partial t}$ can be written as $\sim \frac{u_0}{t_s}$ and the scale of the term $\nu \frac{\partial^2 u}{\partial y^2}$ can be written as $\sim \nu \frac{u_0}{\delta^2}$. This problem has a unique t_s. In the previous problem we can vary t_s at our will but here t_s is governed by $\frac{1}{\omega}$ which is the most important physics of this problem; other part is just trivial mathematics. So if t_s is of the order of $\frac{1}{\omega}$, then $\frac{u_0}{t_s}$ can written as $u_0 \omega$ and $u_0 \omega$ becomes of the order of $\nu \frac{u_0}{\delta^2}$. This implies that δ is of the order of $\sim \sqrt{\frac{\nu}{\omega}}$, it does not vary with time. It is fixed order of δ which is governed by the two parameters ν and ω. Now we can non-dimensionalize governing equation $\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$. u is non-dimensionalized as $\bar{u}=\frac{u}{u_0}$, t is non-dimensionalized as $\bar{t}=\frac{t}{t_s}$ and y is non-dimensionalized as $\bar{y}=\frac{y}{\delta}$ where $\delta=\sqrt{\frac{\nu}{\omega}}$. If we put the relevant scales in the equation $\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$ we will get $\frac{\partial \bar{u}}{\partial \bar{t}}=\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$ because all the coefficients will be absorbed in the non-dimensional parameters, one can easily check that. There are many ways of solving this equation. Here we will use a simple trick to solve this equation. Let us assume $\bar{u}=\mathrm{Im}\left[e^{i\bar{t}} f\left(\bar{y}\right)\right]$. Now question may arise about why we have chosen this particular form. The boundary velocity is $\bar{u}=\sin\bar{t}$ at $\bar{y}=0$. $\sin\bar{t}$ is the imaginary part of $e^{i\bar{t}}$ because $e^{i\bar{t}}$ is equal to $\cos\bar{t}+i\sin\bar{t}$. Then we can adjust the boundary condition by noting that $\bar{u}$ will be equal to $\sin\bar{t}$ if $f\left(\bar{y}\right)$ is equal to 1. So it means the imaginary part of $e^{i\bar{t}}$ should match with the form $\sin\bar{t}$. If the boundary condition was $\bar{u}=\cos\bar{t}$ at $\bar{y}=0$ then we would have assumed $\bar{u}=\mathrm{Re}\left[e^{i\bar{t}} f\left(\bar{y}\right)\right]$ where $\mathrm{Re}\left[e^{i\bar{t}}\right]$ indicates the real part of the function $e^{i\bar{t}}$. Substituting $\bar{u}=\mathrm{Re}\left[e^{i\bar{t}} f\left(\bar{y}\right)\right]$ in $\frac{\partial \bar{u}}{\partial \bar{t}}=\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$ we get imaginary terms on both sides and we take the inner part and do the algebra, i.e. $\frac{\partial \bar{u}}{\partial \bar{t}}=i\,e^{i\bar{t}} f$ and $\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}=f''e^{i\bar{t}}$. Equating these expressions of $\frac{\partial \bar{u}}{\partial \bar{t}}$ and $\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$ we get the differential equation $f''-i\,f=0$. If we take $e^{m\,t}$ as the trial solution to solve this differential equation, we get $f=c_1 e^{\sqrt{i}\,\bar{y}}+c_2 e^{-\sqrt{i}\,\bar{y}}$ where c_1 and c_2 are the constants to be determined using the boundary conditions. If we take that at $\bar{y}\to\infty$, f is finite then the term $c_1 e^{\sqrt{i}\,\bar{y}}$ should not be there. So c_1 should be equal to zero. So, f becomes equal to $c_2 e^{-\sqrt{i}\,\bar{y}}$, i.e. $f=c_2 e^{-\sqrt{i}\,\bar{y}}$ which represents an exponential decay. Now i can be written as $i=e^{i\pi/2}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$. Therefore, $\sqrt{i}$ can be written as $\sqrt{i}=e^{i\pi/4}=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1+i}{\sqrt{2}}$. So, the function f becomes $f=c_2 e^{-\left(\frac{1+i}{\sqrt{2}}\right)\bar{y}}$. Now the evaluation of c_2 is straightforward. The boundary condition $\bar{u}=\sin\bar{t}$ is valid at $\bar{y}=0$. At $\bar{y}=0$ we must have $f=1$ which means $c_2=1$. So the solution of this problem is

$\bar{u} = \mathrm{Im}\left[e^{i\bar{t}} e^{-\left(\frac{1+i}{\sqrt{2}}\right)\bar{y}}\right] = \mathrm{Im}\left[e^{-\frac{\bar{y}}{\sqrt{2}}} e^{i\left(\bar{t}-\frac{\bar{y}}{\sqrt{2}}\right)}\right] = e^{-\frac{\bar{y}}{\sqrt{2}}} \sin\left(\bar{t} - \frac{\bar{y}}{\sqrt{2}}\right)$. So we can notice that the solution is sinusoidal in nature. So the nature is same as present in the boundary condition but there is a phase difference. In the boundary condition it was $\sin\bar{t}$, but in the solution it is $\sin\left(\bar{t} - \frac{\bar{y}}{\sqrt{2}}\right)$ which is dependent on the spatial co-ordinate $\bar{y}$. It is therefore suggested to the students to make a plot of $\bar{u} = e^{-\frac{\bar{y}}{\sqrt{2}}} \sin\left(\bar{t} - \frac{\bar{y}}{\sqrt{2}}\right)$ as a function of $\bar{y}$ and $\bar{t}$ using any kind of software. It will give the students a lot of interesting physical insight. So in this chapter we have discussed about two very important unsteady problems in terms of exact solution of Navier Stokes equation; namely Stokes 1st problem and Stokes 2nd problem.