Advanced Concepts In Fluid Mechanics
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Lecture - 02

Lines of Flow Visualization and Acceleration of Flow

I. Streamlines, Streaklines and Pathlines

We discussed the formal definition of streamline, streakline and pathline in the previous lecture. Now, to obtain a functional form of these lines, we will work out an example.

Prob. – Consider the flow-field $u = \dfrac{x}{t+t_0}$, $v = v_0$, $w = 0$. Find:

(i) Streamline: equation of a streamline passing through a point (x_0, y_0, z_0) at time $t = t_0$ (the subscript '0' is used symbolically and doesn't necessary mean the values are zero)

(ii) Streakline: Equation of a coloured line at time $t = 2t_0$, that is visible as a consequence of dye injection at (x_0, y_0, z_0), starting from $t = t_0$ and continuing till $t = 2t_0$

(iii) Pathline: Locus of a fluid particle that is at (x_0, y_0, z_0) at time $t = t_0$

Soln. –

(i) As discussed in previous lecture, we integrate $\dfrac{dx}{u} = \dfrac{dy}{v}$ to obtain the equation for streamline. Only two terms are there because this is two-dimensional flow-field. Substituting the expressions for u and v in the equation, $\dfrac{dx(t+t_0)}{x} = \dfrac{dy}{v_0}$. Since the streamline at $t = t_0$ is sought, we substitute it in the equation and get $\dfrac{dx}{x} = \dfrac{dy}{2t_0 v_0}$ and integrate to get $y = 2t_0 v_0 \ln(x) + c$, where the constant of integration, c, is obtained by putting $x = x_0$ and $y = y_0$, giving us $y = y_0 + 2t_0 v_0 \ln\left(\dfrac{x}{x_0}\right)$.

(ii) To obtain the equation for the streakline, we follow the following approach. Since streakline is the coloured line created due to injection of dye at a point in the flow-field over a certain period of time, it is actually the locus of the different particles that have passed the injection point at some instance of time. Hence, we consider one such particle P. For this particle, $u_P = \dfrac{dx_P}{dt}$ is the definition of the x-component of its velocity, similar expression exists for the y-component. Since the velocity of a particle at a point is the same as the velocity at that point, we obtain a merger of the Lagrangian and Eulerian descriptions, and therefore have $u = \frac{dx}{dt}$. Substituting the expression for u we have, $\frac{x}{t-t_0} = \frac{dx}{dt}$. This equation is integrated to get the expression for the streakline, $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_i}^{2t_0} \frac{dt}{t+t_0}$. The limits of integration in this equation are crucial. The lower limit for LHS is x_0 as it is the x-coordinate for the point where the dye is injected, and the lower limit of RHS is the time at which the arbitrary particle P passes through the injection point. This time, t_i will be bounded between t_0 and $2t_0$ as that is the duration over which dye is injected. The upper limit of integration for the LHS and RHS are x and $2t_0$ respectively as x is the position of the arbitrary particle P at time $2t_0$. Solving this, we have $\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{3t_0}{t_i + t_0}\right)$. Similarly, the expression for the y-component is $y - y_0 = v_0(2t_0 - t_i)$. Eliminating t_i from these expressions yields the equation for the streakline. The final expression is $\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{3t_0v_0}{3t_0v_0 + y_0 - y}\right)$.

(iii)The approach to obtain the equation for the pathline is the same as the streakline uptil the integration, $\int \frac{dx}{x} = \int \frac{dt}{t+t_0}$. For a pathline, the particle P is not arbitrary, and therefore, the lower limit of integration for the RHS is t_0 rather than t_i. On the other hand, the upper limit of integration for the RHS is an arbitrary time t, because pathline is the locus of the positions occupied by the particle P at different instances of time. Therefore, the integral becomes $\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} \frac{dt}{t+t_0}$. Solving this, we get $\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{t+t_0}{2t_0}\right)$. Similarly, the expression for the y-component is $y - y_0 = v_0(t - t_0)$. Elimiating t from these expressions gives the equation for pathline, which is $\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{y - y_0 + 2t_0v_0}{2t_0v_0}\right)$.

The distinction between the equations for streakline and pathline can be summarized by the statement – in the equation for streakline, the initial time is eliminated, whereas, in the equation for pathline, the final time is eliminated.

While streamlines, streaklines and pathlines are not constrained to be co-incident, under the special case of steady flows, they are co-incident. To clarify, let us consider a particle moving in a steady flow field. The particle assumes the positions P_1, then P_2, then P_3, and then P_4 at infinitesimal durations chronologically. Therefore, the curve joining these points is evidently the pathline. However, since the flow is steady, any other particle at any other instance of time at P_1 is bound to go to P_2, and then P_3, and then P_4. Therefore, the curve joining these four points is also the streakline. Finally, because the velocities are not changing with time at a given location, whenever a particle comes at P_1, it will be guided along the same curve towards P_2. So, curve is such a curve that the tangent to it will always represent the velocity of the flow. So, this line is also the streamline. An artifact of this co-incidence of streamlines and streaklines for steady flow is that when one visualizes a steady flow experiment using a dye injection, the curves seen become visual representations for the streamlines in addition to streaklines. However, this such visualizations should be done with caution as once the flow deviates from being steady, this fortunate co-incidence of streamlines and streaklines vanishes, and what one is seeing are only streaklines.

II. Acceleartion of Flow

We will now discuss the acceleration of flow. Acceleration is the time rate of change of velocity and therefore, it is important to study acceleration whenever we are studying velocity.

From our understanding of high school physics, we can recall that the expression for acceleration $\vec{a}$, is $\vec{a} = \frac{d\vec{v}}{dt}$. In the context of fluid mechanics, it is important to recognize that this expression for acceleration is applicable when the velocity is described as per the Lagrangian decription. However, in a flow field, the expression for velocity that is given is as per the Eulerian description. Hence, it becomes crucial to derive the expression for acceleration of a fluid particle in terms of the velocity field given in a Eulerial description.

To derive this expression, consider a particle P which at a point (x, y, z) and has velocity $\overrightarrow{V_P}$. This velocity is same as the velocity in the flow field at the point (x, y, z) at time t, $\vec{v}$, which is a function of the position (x, y, z) and time t, i.e. $\overrightarrow{V_P} = \vec{v}(x, y, z, t)$. After some time Δt, the particle flows to the point $(x+\Delta x, y+\Delta y, z+\Delta z)$, and its velocity becomes $\overrightarrow{V_P} + \Delta\overrightarrow{V_P}$. This velocity will be same as the velocity in the flow field at the point $(x+\Delta x, y+\Delta y, z+\Delta z)$ at time $t+\Delta t$, i.e. $\overrightarrow{V_P} + \Delta\overrightarrow{V_P} = \vec{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$. The acceleration of particle P at time t can be obtained by considering an infinitesimally small instance of time in the future, i.e. in the limit $\Delta t \to 0$. Represented by $\overrightarrow{a_P}$, the acceleration of the particle P is

$$\overrightarrow{a_P} = \lim_{\Delta t \to 0} \frac{\vec{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - \vec{v}(x, y, z, t)}{\Delta t} .$$

Using the Taylor series expansion of $\vec{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t)$, which is

$$\vec{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) = \vec{v}(x,y,z,t) + \frac{\partial \vec{v}}{\partial t}\Delta t + \frac{\partial \vec{v}}{\partial x}\Delta x + \frac{\partial \vec{v}}{\partial y}\Delta y + \frac{\partial \vec{v}}{\partial z}\Delta z + H.O.T\ .$$

$H.O.T$ are the higher order terms, that are negligible in the limit $\Delta t \to 0$. Substituting the Taylor series expansion in the expression for $\overrightarrow{a_P}$, and recognizing that $\lim\limits_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = u_P$, $\lim\limits_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = v_P$ and $\lim\limits_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = w_P$, $\overrightarrow{a_P} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x} u_P + \frac{\partial \vec{v}}{\partial y} v_P + \frac{\partial \vec{v}}{\partial z} w_P$.

In the limit of $\Delta t \to 0$, original position of particle and new position of particle get infinitesimally close, and so, the Lagrangian and Eulerian descriptions converge. Therefore, dropping the subscript P, the expression for acceleration in the Eulerian description is

$$\overrightarrow{a} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}\ .$$

In this expression, there are clearly two different types of expressions. The first expression, $\frac{\partial \vec{v}}{\partial t}$, is of the first type and it gives information about the steadiness or unsteadiness of flow. It is the acceleration due to change of velocity with time at a given location in the flow field. It is called as unsteady or temporal component. The next three expressions are of the second type, which signify the acceleration due to difference in velocities of two locations in flow field that the particle flows between. This is called as convective component of acceleration. The net acceleration is the combination of these two. Combined together, this spatiotemporal acceleration is commonly denoted by $\frac{D\vec{v}}{Dt}$, with $\frac{D}{Dt}$ commonly called 'Total Derivative'.

In vector notation, this acceleration is written as

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\ .$$

The advantage of writing in vector notation is that it applies to non-cateresian coordinate systems as well, like cylindrical coordinate system and spherical coordinate system.

The next lecture will discuss deformation of fluids, which is very different from deformation of solids.