Advanced Concepts In Fluid Mechanics

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Lecture - 17

Exact Solution of Navier Stokes Equation: Example of Fully Developed Flow

In the previous chapter, the Navier Stokes equation and its derivation were discussed. In this chapter, the main focus is to identify some physical problems for which the exact solution of the Navier Stokes equation is possible. These problems are limited in number since most of the practical problems are essentially non-linear. However, there are certain cases where the practical problems are also linear in nature.

The first example under consideration is the pressure-driven internal flow. There are two parallel plates. This is a simple but very elegant assumption of internal flow. Internal flow means that there is a confined passage within which the fluid is flowing which may be driven by a pressure gradient. The flow may be closed from all directions typically observed in the case of a rectangular duct or circular duct. If the width of the rectangular duct is much larger than the height, then it can be treated as two parallel plates because of negligible end effects and can be considered as a parallel plate channel.

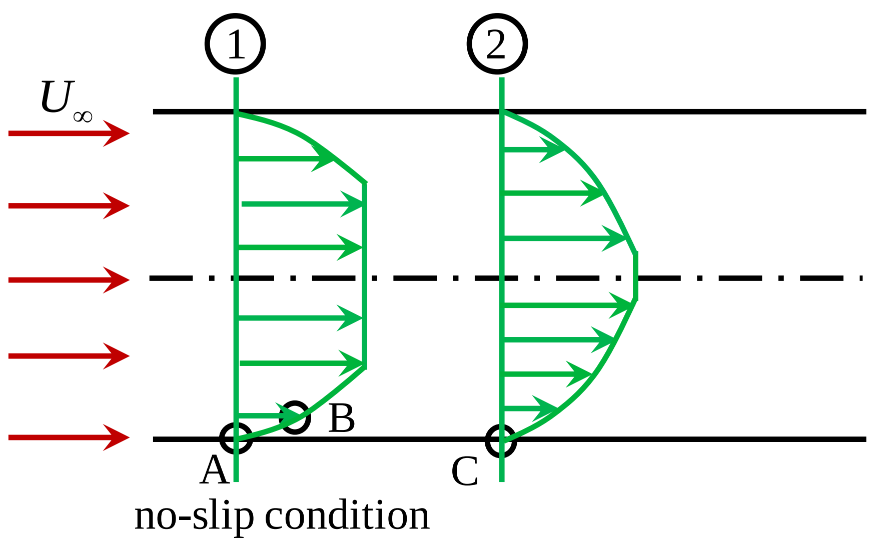

Fig 1. Schematic of the flow through a parallel plate channel.

Here, figure 1 represents a parallel plate channel where the dash-dot line indicates the center-line of the channel. The fluid enters from the far stream with a velocity U_∞. When the fluid encounters the solid boundary (shown by section 1 of figure 1), because of the viscous effect, the fluid will come to a standstill; this is known as no-slip boundary condition (applicable at point A). The velocity somewhere above this location (i.e. at point B in figure 1) is greater than zero but not the maximum one. In this way, it will come to a state where the velocity will become maximum and beyond which it will not change further. Since the problem is symmetric, no-slip condition along with other boundary conditions are also applicable on the other side of the channel.

Now, as we go to a section (section 2) that is a little bit advanced, the wall effects become more pronounced. Since we enter more into the channel, greater is the depth up to which the viscous effects are felt and accordingly the velocity gradient should exist up to that depth. Now we draw the locus of the points within which the wall effect creates the velocity gradient, this locus (violet-colored line in figure 2) is called the edge of the boundary layer. The boundary layer is the layer close to the solid boundary where the viscous effects are felt thereby giving rise to the velocity gradient.

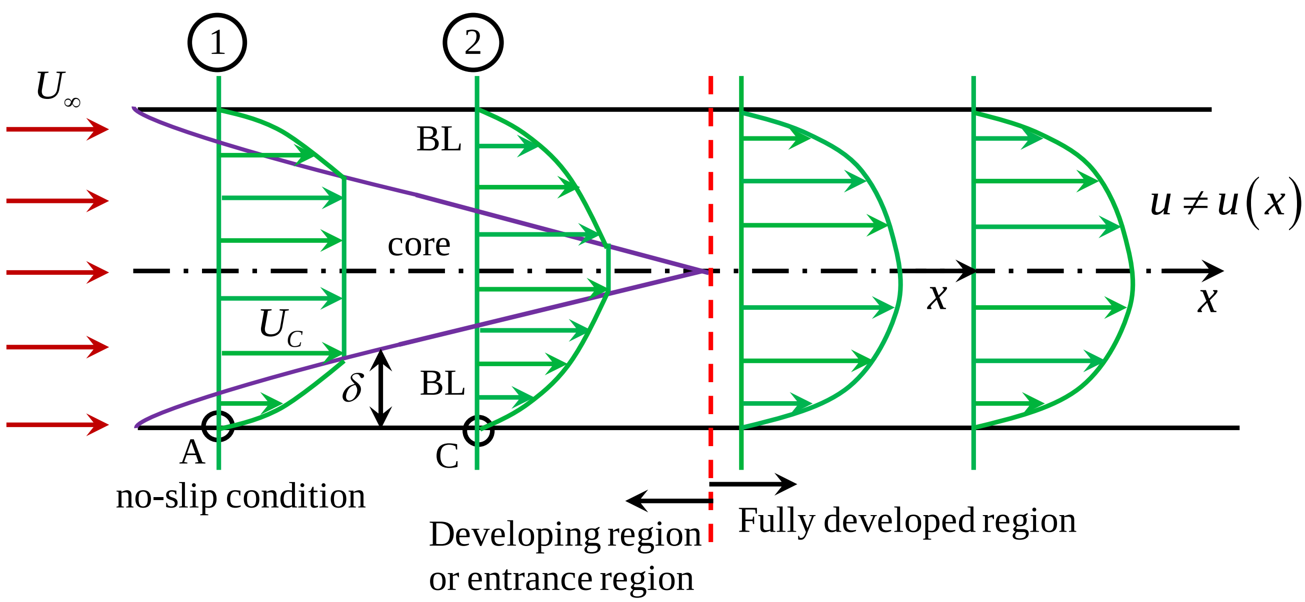

Fig 2. Different regions for the flow through a parallel plate channel.

The region occupied by the violet-colored lines is known as the core region while the outside region is known as the boundary layer region. As shown in figure 2, the boundary layers merge with each other at somewhere close to the section 2 (this position is located by the red colored dotted line in figure 2). Up to this location, the boundary layers are growing and it is known as developing region. Within the developing region, the core fluid does not understand the presence of the solid boundary but beyond this region, the entire fluid understands the effect of the solid boundary. Beyond the developing region, the flow has become fully developed which means that the velocity profile does not change in the axial direction anymore.

So, the region left to the red colored dotted line is called developing region or entrance region and the region right to the dotted line is called fully developed region. The fully developed region is a region where the velocity u is no longer a function of *x*-coordinate. For the sake of understanding, we now focus to obtain some plots like the variations of core velocity $\left(U_C\right)$, wall shear stress $\left(\tau_w\right)$ and pressure $\left(p\right)$ with axial coordinate *x*.

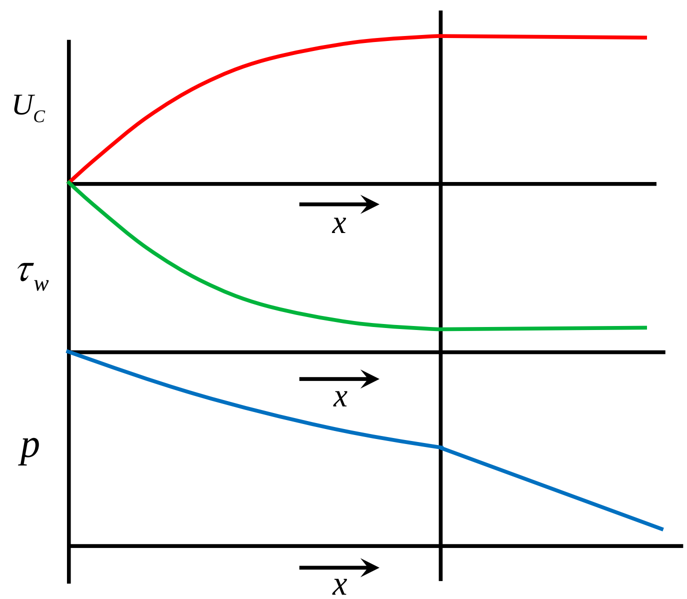

Fig 3. Variations of core velocity $\left(U_C\right)$, wall shear stress $\left(\tau_w\right)$ and pressure $\left(p\right)$ as a function of axial coordinate *x*.

Let us first consider two sections 1 and 2 as depicted in figure 2. Now the question is that, between these two sections, where the core velocity is more. The flow rates are constant in these two sections and if we assume the density of the fluid (ρ) to be constant, then the volumetric flow rate is also constant. This volume flow rate can be obtained by integrating the velocity profile across the cross-section.

So, as we move from section 1 to section 2, the boundary layer becomes thicker which means that there is a thicker region over which the fluid is decelerated. In order to maintain the same flow rate, the velocity should be more in the region where the fluid is accelerated since the integral of these two sections is the same. That means the core velocity in section 2 is greater than the core velocity in section 1 until it becomes fully developed and then the core velocity does not change with *x*. The axial variation of the core velocity $\left(U_C\right)$ is shown by the red-colored solid line in figure 3.

Now we focus on the wall shear stress. Regarding this, let us first consider the fully developed region. The deviatoric stress is what that dictates Newton's law of viscosity, i.e. $\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$. At the wall, there is no *v*-component which is known as no penetration boundary condition. Now *u* is 0 at the wall because of the no-slip condition while *v* component is zero because of no penetration condition which means that the fluid cannot penetrate through the wall. Since there is no *v* component, $\frac{\partial v}{\partial x}$ is 0, $\tau_{wall} = \tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w$, the fully developed velocity u does not change with *x*, the wall shear stress also does not change with *x*. So, the wall shear stress in the fully developed region is axially invariant.

Now the question arises about the variation of the same with *x* in the developing region. Form definition, the wall shear stress is of the order of ~ $\mu\frac{U_C}{\delta}$ where U_C is the core velocity and δ is the boundary layer thickness. The interesting thing to observe is that as one moves along the entrance region U_C increases and δ also increases. So, now, we need to determine the ratio $\frac{U_C}{\delta}$ in order to check whether it is increasing or decreasing. When the fluid just enters the channel, δ is tending towards zero while U_C is finite. So the resulting shear stress will be very large. Understandably, the wall shear stress should come down asymptotically from such a large value and it cannot blow up. This variation with *x* is depicted by the green-colored solid line in figure

3. Now, to understand the pressure variation with *x*, let us consider a fluid element in the fully developed region where τ_w indicates the wall shear stress on both walls.

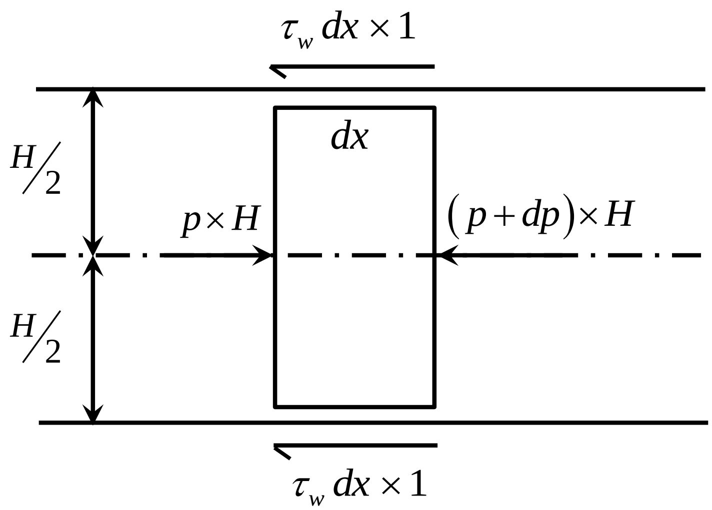

Fig 4. Force balance across a differential element of length *dx*.

Let us take the differential element of length '*dx*'. The viscous force acting is denoted by τ_w multiplied by the width of the element (*W*) which is chosen as unity. The total height of the channel is *H* (shown by two equal *H*/2 lengths in figure 4). The force due to pressure can be quantified as $p \times H \times 1$ on the left-hand side of the element and $\left(p+dp\right)\times H \times 1$ on the right-hand side of the element. We are assuming that *p* is not a function of the transverse coordinate *y* and is a function of *x* only. One can justify this assumption using the *y*-component of the momentum equation, where one can show that the variation of *p* with *y* is indeed 0.

For fully developed flow, since there is no acceleration, the velocity profile is not changing which means all acting forces are balanced. So, the force balance gives $pH-\left(p+dp\right)H-2\tau_w\,dx=0$ where the wall shear stress $\left(\tau_w\right)$ is given by $\tau_w = \frac{H}{2}\frac{dp}{dx}$. So, if τ_w remains constant, $\frac{dp}{dx}$ also remains constant which means the variation of *p* with *x* will be linear in nature in fully developed region, while in the developing region, it is non-linear in nature. This variation of pressure (*p*) with *x* is shown by the blue colored solid line in figure 3. This kind of qualitative depiction without getting into much of mathematics, but getting a physical picture of how the basic variables are varying is very important to develop in physics and engineering. The rest of the chapter is directed towards the derivation of the velocity profile in the fully developed region which is depicted in figure 2 as parabolic distribution (the rightmost velocity profile of figure 2).

Fully developed flow between two parallel plates (Plane Poiseuille flow)

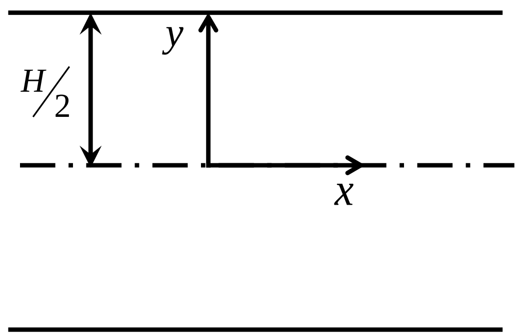

Figure 5. Geometry of the fully developed flow between two parallel plates.

Now we consider fully developed flow between two parallel plates; this is called plane Poiseuille flow. So, let us assume two dimensional plus incompressible flow. So, we have considered two parallel plates as shown in figure 5, the centerline is *x*-axis and the transverse direction is the y-axis. The incompressibility equation is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

For fully developed flow, u is not a function of *x*, $\frac{\partial u}{\partial x}=0$ which also means that *v* is not a function of *y* since $\frac{\partial v}{\partial y}$ is 0. At the walls, i.e. at $y=\pm\frac{H}{2}$, *v* is 0 because of no penetration condition (not to be confused with the no-slip condition). So, this means *v* is zero for all *y*. Now, the *x*-component of the Navier Stokes equation is written as

x-component:
$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

This equation (2) is just a traditional way of writing the Navier Stokes equation, there is no body force term.

Then we can focus on the y-component of the Navier Stokes equation for which all of the previous assumptions remain valid. We assume the flow to be steady, so $\frac{\partial u}{\partial t}$ becomes 0. The term $\frac{\partial u}{\partial x}$ becomes zero for fully developed flow and v becomes zero as a corollary of fully developed flow provided there are no holes at the wall. The situation will be different if there are holes even if the fully developed condition is satisfied. With these assumptions, the simplified form of equation (2) is written below

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{3}$$

where $\frac{\partial^2 u}{\partial y^2}$ can be replaced as $\frac{d^2 u}{dy^2}$ since u is not a function of x and becomes a function of y only for fully developed flow. So when there are no holes on the wall, v is zero everywhere; most of the terms in the y-component of the momentum equation involves v and thus become zero. The general form of the y-momentum equation is given below

y-component:
$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{4}$$

With the aforesaid considerations, the simplified y-momentum equation becomes $-\frac{\partial p}{\partial y} = 0$ which means that p is not a function of y. So, p becomes a function of x only. Hence, the $\frac{\partial p}{\partial x}$ term in the x-momentum equation can be replaced by $\frac{dp}{dx}$. Therefore, the simplified x-momentum equation becomes

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 u}{dy^2} \tag{5}$$

which means $\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$. Physically what it means is that the pressure force due to pressure gradient is balanced exactly by force due to viscous effect and that is how the fluid is not accelerating. Integrating equation (5) once with respect to y gives

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx} y + c_1 \tag{6}$$

while integrating equation (6) results in the following expression of the velocity distribution

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + c_1 y + c_2 \tag{7}$$

Since the problem is symmetric, one can take only half of the domain and solve it easily. Now, we employ the boundary conditions in order to obtain the integration constants c_1 and c_2.

At the channel centerline, i.e. at $y = 0$, we have $\frac{du}{dy} = 0$. Since it is symmetric, u becomes maximum at the channel centreline, therefore velocity gradient vanishes and thus $c_1 = 0$. At $y = \frac{H}{2}$, $u = 0$ because of no-slip condition which means $0 = \frac{1}{2\mu}\frac{dp}{dx}\left(\frac{H}{2}\right)^2 + c_2$, so, c_2 becomes $c_2 = -\frac{1}{2\mu}\left(\frac{H}{2}\right)^2\frac{dp}{dx}$. Substituting c_1 and c_2, the final form of the velocity profile is given by

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left[y^2 - \left(\frac{H}{2}\right)^2\right] \tag{8}$$

Here, y is always less than $\frac{H}{2}$ and $\frac{dp}{dx}$ is negative because the fluid is being driven by a pressure gradient from higher pressure to lower pressure. So, both $\frac{dp}{dx}$ and $\left[y^2 - \left(\frac{H}{2}\right)^2\right]$ terms are negative and the combination of these two results in the positive value of the velocity. Sometimes the pressure gradient is not measured, but the flow rate is measured. So, it is possible to express this in terms of the average velocity $(\bar{u})$ which is defined as $\bar{u} = \frac{\int_{-H/2}^{H/2} u\, dy \times W}{H \times W}$. Conceptually the average velocity is the equivalent velocity which could have prevailed had the velocity profile be uniform and the same flow rate would have been maintained. Here, the numerator $\int_{-H/2}^{H/2} u\, dy \times W$ indicates the flow rate where the area can be given by the height H multiplied by unit width W. The width terms get canceled from the numerator and denominator and substituting this expression of average velocity in the velocity profile one can rewrite the velocity distribution as

$$\frac{u}{\bar{u}} = \frac{3}{2}\left(1 - \frac{y^2}{H^2}\right) \tag{9}$$

Equation (9) is a parabolic distribution which is the expression of the fully developed velocity profile for flow between two parallel plates (also known as the so-called plane Poiseuille flow).

In summary, some exact solutions of the Navier Stokes Equation are discussed which will be continued in the next chapter.