Advanced Concepts In Fluid Mechanics
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Lecture - 15
Navier Stokes Equation - Part1

Navier-Stokes equation

We recapitulate the Cauchy's equation of motion that we have arrived at in the previous lecture.

$$\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right]=\frac{\partial \tau_{ij}}{\partial x_j}+b_i \qquad (1)$$

Along with this equation, we have the equation for mass conservation, expressed as,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)=0 \qquad (2)$$

Note that there are three components of equation (1) for i=1,2,3 and one component of equation (2) (as the index j gets summed over). Therefore, we have a total of four governing equations.

In fluid mechanics, the typical physical variable of concern is the velocity of the fluid, which is needed in multiple scenarios. However, to solve for the fluid velocity from the governing equations and boundary conditions, we are required to solve for other system variables as well. To ascertain all of them, we take a look at equations (1) and (2), which are the governing equations for flow. We can see that we need to solve for the density ρ, the three components of velocity v_i (i=1,2,3), and the six **independent** components of stress tensor τ_{ij} (i=1,2,3, j=1,2,3), i.e. a total of 10 unknowns. Note that there is another governing equation, the equation for conservation of angular momentum, which has three components for the three planes x-y, y-z, and x-z. However, we have utilized these in the previous lecture to bring the number of unknown stress tensor components from nine to six by establishing that three of the stress tensors are dependent on (more specifically, equal to) their converses, and therefore, there are only six independent components of the stress tensor.

Hence, we have ten unknowns and four equations to solve them from. There is a deficit of six equations to close the problem mathematically. The additional six equation we require are obtained from the **constitute formulation** for the fluid, i.e. the appropriate expressions for the stress tensor components τ_{ij}.

Hence, in this lecture, we will focus on obtaining the appropriate expression for τ_{ij}. We decompose the stress τ_{ij} into two parts – the first part corresponds to the stress that exists in the fluid when it is devoid of any motion, and is aptly called the hydrostatic component; the second part corresponds to the stress emerging out of the deformation occurring in the fluid due to motion, and is termed as the deviatoric component.

$$\tau_{ij} = \tau_{ij}^{hyd} + \tau_{ij}^{dev} \tag{3}$$

Clearly, the hydrostatic component of the stress tensor, τ_{ij}^{hyd}, will simply be the pressure in the fluid (we will get to the formal expression later in the lecture).

On the other hand, we have to formulate the deviatoric component of stress tensor, τ_{ij}^{dev}, as a mathematical expression for flow. Based on experimental observations for many fluids from classical times in fluid dynamics, fluid dynamicists have determined that the deviatoric stress in a fluid will depend on the rate of strain of the fluid. Therefore, we take a look at the general expression for the rate of strain in a fluid, $\frac{\partial v_k}{\partial x_l}$. This expression can be elaborated into the form

$$\frac{\partial v_k}{\partial x_l} = \frac{1}{2}\left[\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right] + \frac{1}{2}\left[\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right] \tag{3}$$

Upon examination, we can observe that the first term in the RHS of equation (3) corresponds to rate of deformation of a fluid element due to motion whereas the second term corresponds to rigid body rotation of the fluid element. Since rigid body rotation is not expected to contribute to the deviatoric stresses emerging in the fluid, we express just the first term as,

$$e_{kl} = \frac{1}{2}\left[\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right] \tag{4}$$

and the deviatoric component of stress tensor will depend on e_{kl}. Note that like τ_{ij}, e_{kl} also has nine components (three for *k* times three for *l*). Therefore, e_{kl} is also a second order tensor. Also, note that like τ_{ij}, e_{kl} also has only six independent components, since $e_{kl} = e_{lk}$ – however, note that this symmetry occurs simply because of the definition of e_{kl} (equation (4)) and we don’t have to resort to any governing equation to arrive at this symmetry. Lastly, also note that as we will see later, τ_{ij}^{hyd} is non-zero only for *i*=*j*, and therefore

$$\tau_{ij} = \tau_{ji} \Rightarrow \tau_{ij}^{dev} = \tau_{ji}^{dev} \tag{5}$$

Since both the deviatoric component of the stress tensor τ_{ij}^{dev}, and the deformation-causing strain rate tensor, e_{kl}, are second-order tensors, the most general form of the dependence of the former on the latter will include a fourth order tensor, c_{ijkl}, which has 81 components (three for *i* times three for *j* times three for *k* times three for *l*).

$$\tau_{ij}^{dev} = c_{ijkl} e_{kl} \tag{6}$$

While c_{ijkl} in principle has 81 components, we require significantly lesser number of variables practically always. By appealing to the symmetry of stress and strain rate tensors, and isotropy and homogeneity of the fluid, we are able to reduce the c_{ijkl} tensor to have only two independent components, as we shall see ahead.

We first invoke the isotropy of the fluid. Isotropy implies that any material property is direction independent. In other words, changing the orientation of co-ordinate axes (i.e. rotating the co-ordinate axes) arbitrarily should not alter the tensor c_{ijkl}. To formally express this in mathematical terms, we define the scalar *s* as the scalar we obtain by transforming under the tensor c_{ijkl}, the fourth order tensor given as $A_i B_j C_k D_l$. Here, A_i, B_j, C_k and D_l are four arbitrary vectors, each of which donate one of the indices *i, j, k* and *l*. Since the magnitude of and the angle between any two vectors, and therefore the dot product (which is simply the product of the magnitude and cosine of the angle), remain conserved under rotation, if we define the scalar *s* to be the generalized summation of the different combinations of the dot product of the four vectors A_i, B_j, C_k and D_l, it will remain the same under rotation. That is,

$$s = \alpha\left(A_i B_i\right)\left(C_k D_k\right) + \beta\left(A_i C_i\right)\left(B_j D_j\right) + \gamma\left(A_i D_i\right)\left(B_j C_j\right) = c_{ijkl} A_i B_j C_k D_l \tag{7}$$

Note that generally, α, β and γ can vary spatially in the fluid bulk. However, for a homogeneous fluid, α, β and γ are constants.

Now, we utilize the Kronecker-delta function δ_{ij} for proceeding ahead with equation (7). δ_{ij} is defined as $\delta_{ij} = 1$ if *i* = *j*, and $\delta_{ij} = 0$ if $i \neq j$. This property of δ_{ij} helps us to write any variable M_i as $M_i = M_1\delta_{i1} + M_2\delta_{i2} + M_3\delta_{i3} = M_j\delta_{ij}$. Using this property, we write equation (6) as,

$$\begin{aligned}
&s = \alpha\left(A_i B_j \delta_{ij}\right)\left(C_k D_l \delta_{kl}\right) + \beta\left(A_i C_k \delta_{ik}\right)\left(B_j D_l \delta_{jl}\right) + \gamma\left(A_i D_l \delta_{il}\right)\left(B_j C_k \delta_{jk}\right) = c_{ijkl} A_i B_j C_k D_l \\
&\Rightarrow s = \left[\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\right] A_i B_j C_k D_l = c_{ijkl} A_i B_j C_k D_l \\
&\Rightarrow c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}
\end{aligned} \tag{8}$$

Now, since both τ_{ij}^{dev} and e_{kl} are symmetric in *i* and *j*, and since c_{ijkl} is defined to transform e_{kl} to τ_{ij}^{dev}, c_{ijkl} should also be symmetric in *i* and *j*. Therefore, interchanging *i* and *j* in the expression for c_{ijkl} should give us the same expression. This implies $\beta = \gamma$, i.e.

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{9}$$

We now proceed with writing the transformation of e_{kl} to τ_{ij}^{dev},

$$
\begin{aligned}
&\tau_{ij}^{dev} = c_{ijkl} e_{kl} \\
&\Rightarrow \tau_{ij}^{dev} = \left[\alpha \delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right] e_{kl} \\
&\Rightarrow \tau_{ij}^{dev} = \alpha \delta_{ij}\delta_{kl} e_{kl} + \beta\left(\delta_{ik}\delta_{jl} e_{kl} + \delta_{il}\delta_{jk} e_{kl}\right) \qquad (10) \\
&\Rightarrow \tau_{ij}^{dev} = \alpha \delta_{ij} e_{kk} + \beta\left(e_{ij} + e_{ji}\right) \\
&\Rightarrow \tau_{ij}^{dev} = \alpha e_{kk}\delta_{ij} + 2\beta e_{ij}
\end{aligned}
$$

Now, β represents the dynamic viscosity of the fluid and is thus replaced with μ - note that for a unidirectional shear flow, equation (10) simply converts to $\tau = \beta \dfrac{dv_1}{dx_2}$, which is the expression for the corresponding shear stress in terms of shear strain rate (also called Newton's law of viscosity) that is taught as the preliminary introduction to viscosity in high school physics, with the co-efficient expressed as μ instead of β. On the other hand, α is conventionally replaced with λ, which is called the second co-efficient of viscosity. Therefore, we finally have the expression for deviatoric stress in terms of strain rate as

$$\tau_{ij}^{dev} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \qquad (11)$$

In equation (11), if $i=j$, we obtain the expression for stress tensor that corresponds to the volumetric change of fluid, and is dependent on both λ and μ. On the other hand, if $i \neq j$, we obtain the stress tensor corresponding to the shape change of fluid, and is dependent only on μ. The overall deformation of the fluid is combination of volumetric change and the shape change. Lastly, note that for a fluid having constant density with time and over space, the conservation of mass (equation (2)) implies e_{kk} is zero, implying the term with λ in equation (11) becomes zero and therefore, λ is inconsequential.

Note that in the derivation of equation (11), we had assumed (just before equation (3)) that the deviatoric component of stress depends linearly on the strain rate of the fluid (equivalently spatial gradient of fluid velocity) and doesn't depend on anything else. Any fluid that satisfies these assumptions is called as 'Newtonian fluid'. While the vast majority of fluids we encounter in fluid mechanical studies are Newtonian, there are exceptions that do not satisfy these assumptions. Examples are power-law fluids, for which deviatoric stress varies with some power of the strain-rate (the power index is simply unity for Newtonian), and Bingham plastics, which are characterized by an additional constant term in the expression for deviatoric stress. However, the fundamentals in the discipline of fluid mechanics pertains to Newtonian fluid and non-Newtonian fluids are studied on top of it.

Lastly, we recall that we still have to express τ_{ij}^{hyd} mathematically. As already mentioned, τ_{ij}^{hyd} is simply because of the pressure in the fluid. Since by definition, pressure on any (real or imaginary) surface in the fluid acts normal to it, is independent of the direction (i.e it is the same along x_1, x_2 or x_3) and is compressive in nature (i.e. positive when compressive), the expression for τ_{ij}^{hyd} is simply

$$\tau_{ij}^{hyd} = -p\delta_{ij} \tag{12}$$

In equation (12), the negative sign represents that pressure is positive when it is compressive, the fact that *p* doesn't have a subscript represents that it is same along any direction, and the Kronecker-delta represents that it acts along the normal direction only. Hence, substituting from equations (11) and (12) into equation (3), we have the expression for stress tensor in terms of the strain rate tensor and the fluid pressure as,

$$\tau_{ij} = -p\delta_{ij} + \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{13}$$

With equation (13) as the constitutive model for Newtonian fluids, the number of unknowns reduces from ten to five, as the six stress tensor components are expressed in terms of fluid velocity and pressure, but pressure emerges as another unknown. Hence, we are still short of one equation. This additional equation is obtained from the equation of state, which acts as the constitutive equation for fluid density.