Advanced Concepts In Fluid Mechanics

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Lecture - 14
Cauchy / Navier Equation

<u>Viscous Forces in Reynolds Transport Theorem (Continued)</u>

We recapitulate the expression for traction vector as we had obtained in the last lecture,

$$T_1^{\eta} = \tau_{11}\eta_1 + \tau_{21}\eta_2 + \tau_{31}\eta_3$$
$$T_2^{\eta} = \tau_{12}\eta_1 + \tau_{22}\eta_2 + \tau_{32}\eta_3$$
$$T_3^{\eta} = \tau_{13}\eta_1 + \tau_{23}\eta_2 + \tau_{33}\eta_3 \quad (1)$$

collectively represented as

$$T_i^{\eta} = \sum_{j=1}^{3} \tau_{ji}\eta_j$$

Note that we have used the (x_1, x_2, x_3) notation for the co-ordinate system rather than (x,y,z) as this assists us with writing the traction vector in the compacted summation form. To make the notation more compact, Einstein suggested dropping the summation sign whenever the summation index is repeated in the summed-over expression – for such a case, the repetition of the index is defined to implicitly imply summation. Therefore, in Einstein notation, the expression for traction vector is simply,

$$T_i^{\eta} = \tau_{ji}\eta_j \quad (2)$$

Now, before proceeding with the Reynold's equation for linear momentum conservation, we first take an assessment of the conservation of angular momentum conservation in the fluid as it is expected to give us some information regarding the stress tensor. To assess this angular momentum conservation however, we do not resort to the Reynold's transport theorem expression for angular momentum conservation. Rather, we consider the planar rectangular elemental control volume presented in Figure 1, which is oriented in the x-y plane.

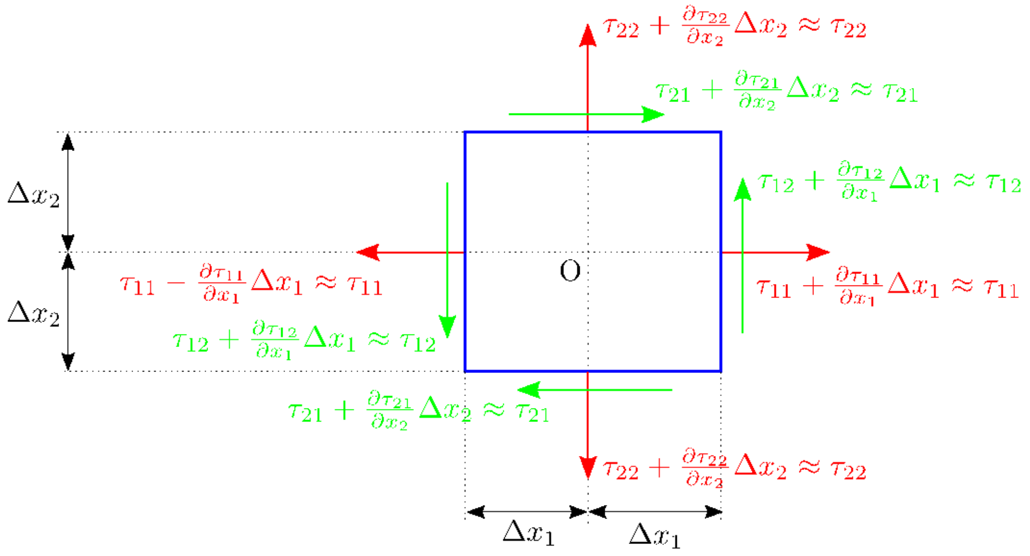

Figure 1: Planar elemental control volume in *x*-*y* plane of unit width into the plane, the approximate expressions in the limiting case of Δx_1 and Δx_2 being infinitesimally small are presented on the right for each expression after '$\approx$'

All the stresses (i.e. the surface forces per unit area) are presented in the figure. We now consider the moment on the elemental control volume about the point O. Only the surface stresses in the colour green contribute to this moment as the other stresses pass through the point O. Therefore, the sum total of moment about the point O is

$$M_O = \left(\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}\Delta x_1\right)\Delta x_1 \Delta x_2 + \left(\tau_{12} - \frac{\partial \tau_{12}}{\partial x_1}\Delta x_1\right)\Delta x_1 \Delta x_2 - \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_2\right)\Delta x_2 \Delta x_1 - \left(\tau_{21} - \frac{\partial \tau_{21}}{\partial x_1}\Delta x_2\right)\Delta x_2 \Delta x_1 = 2\Delta x_2 \Delta x_1 \left(\tau_{12} - \tau_{21}\right) \tag{3}$$

In the $\Delta x_i \Delta x_j$ after each of the brackets in the middle term in equation (3), the first term $\Delta x\,(\Delta x_i)$ is the distance of the force from O and the second $\Delta x\,(\Delta x_j)$ is the length of the side.

Now, the net moment on the elemental control volume should equal the angular momentum of the control volume, i.e,

$$I\frac{\partial \omega}{\partial t} = \frac{\rho \Delta x_1 \Delta x_2}{12}\left[\left(\Delta x_1\right)^2 + \left(\Delta x_2\right)^2\right]\frac{\partial^2 \theta}{\partial t^2} = M_O \tag{4}$$

From equations (3) and (4), we have (upon equating M_O from each equation and then dividing by $2\Delta x_1 \Delta x_2$),

$$\tau_{12} - \tau_{21} = \frac{\rho}{24}\left[\left(\Delta x_1\right)^2 + \left(\Delta x_2\right)^2 \right] \frac{\partial^2 \theta}{\partial t^2} \quad (5)$$

Considering the elemental control volume to be arbitrarily small, the RHS in equation (5) vanishes and we have,

$$\tau_{12} = \tau_{21} \quad (6)$$

Similar analysis on elemental control volumes oriented along *x-z* and *y-z* planes can be done. Collectively from these analyses, we have,

$$\tau_{ij} = \tau_{ji} \quad (7)$$

Therefore, out of the nine components of the stress tensor, only six are independent due to the symmetry obtained in equation (7).

However, a word of caution is required here. In obtaining equation (7), we have assumed that there are no other moments on the elemental control volume in figure 1 apart from the ones due to the boundary surface stresses. Examples where such is not the case are fluid with rotating particulate matters and micropolar fluids. In technical terminology, such scenarios, and any other cases where couples apart from boundary surface stresses emerge, the fluid is said to consist of 'body couples', which need to be accounted for and which break the symmetry observed in equation (7).

As a consequence of (7), equation (2) becomes,

$$T_i^{\eta} = \tau_{ji}\eta_j = \tau_{ij}\eta_j \quad (8)$$

Now, we recall the equation corresponding to Reynold's transport theorem for conservation of linear momentum,

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}(\vec{v}_r \cdot \vec{\eta}) dA \quad (9)$$

Following the convention in fluid mechanics, we interchange the LHS and RHS of equation (9) and we replace the expression for $\sum \vec{F}$ from equation (2) of the last lecture.

$$\frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}(\vec{v}_r \cdot \vec{\eta}) dA = \sum \vec{F}_{surface} + \sum \vec{F}_{body} = \int_{CS} \vec{T}^{\eta} dA + \int_{CV} \vec{b} dV \quad (10)$$

Now, converting equation (8) from index notation to vector notation, we get

$$T_i^{\eta} = \tau_{ij}\eta_j \Rightarrow \vec{T}^{\eta} = \left(\tau_{i1}\overrightarrow{x_1} + \tau_{i2}\overrightarrow{x_2} + \tau_{i3}\overrightarrow{x_3}\right) \cdot \left(\eta_1\overrightarrow{x_1} + \eta_2\overrightarrow{x_2} + \eta_3\overrightarrow{x_3}\right) = \underline{\underline{\tau}} \cdot \vec{\eta} \quad (11)$$

Substituting from equation (11) into equation (10) we have

$$\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV+\int_{CS}\rho\vec{v}(\vec{v}_r\cdot\vec{\eta})dA=\int_{CS}\underline{\underline{\tau}}\cdot\vec{\eta}dA+\int_{CV}\vec{b}dV \qquad (12)$$

Now, we apply the divergence theorem on the second term of the LHS and the first term of the RHS of equation (12) to get,

$$\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV+\int_{CV}\nabla\cdot(\rho\vec{v}\vec{v}_r)dV=\int_{CV}\nabla\cdot\underline{\underline{\tau}}dV+\int_{CV}\vec{b}dV \qquad (13)$$

Note that the objective of defining a stress tensor and of expressing the traction vector as presented in equation (11), particularly in the purview of fluid mechanics, is to eventually apply the divergence theorem (as has been done in obtaining equation (13)). And the reason for wanting to apply divergence theorem is so that the different integrals in the integral equation for momentum conservation convert into integrals over the volume of the control volume – this assists in getting rid of the integral later, as we shall see.

Now, to simplify the analysis ahead, we assume the control volume to be non-deforming (i.e. $\frac{\partial}{\partial t}$ moves into the integral in the first term of LHS of equation (13) without requiring any modifications) as well as stationary (i.e. $\vec{v}_r=\vec{v}$). With these assumptions, equation (13) becomes

$$\int_{CV}\left[\frac{\partial}{\partial t}(\rho\vec{v})+\nabla\cdot(\rho\vec{v}\vec{v})-\nabla\cdot\underline{\underline{\tau}}-\vec{b}\right]dV=0 \qquad (14)$$

Note that we could consider control volume to be deforming as well as non-stationary (even accelerating). In such a situation, we would have to consider the appropriate correction terms to the first integral on LHS and to the body force term on the RHS to account for the control volume deformation and acceleration respectively. However, we would still arrive at the same result as equation (15) ahead. Therefore, the simplifying assumptions of non-deforming and stationary control volume have been made to avoid unneeded tedium.

Now, since the control volume is arbitrarily shaped, the integral in equation (14) will be zero only when the integrand itself is zero. Therefore, we get the equation,

$$\frac{\partial}{\partial t}(\rho\vec{v})+\nabla\cdot(\rho\vec{v}\vec{v})=\nabla\cdot\underline{\underline{\tau}}+\vec{b} \qquad (15)$$

Equation (15) can be expressed using index notation as well, as below,

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial}{\partial x_j}(\rho v_i v_j)=\frac{\partial\tau_{ij}}{\partial x_j}+b_j \qquad (16)$$

Furthermore, equation the LHS of equation (16) can be expanded as

$$v_i\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)\right]+\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right] \tag{17}$$

The left square-bracketed term in equation (17) is zero due to mass conservation. We therefore arrive at an alternate form of equation (16) as

$$\rho\left[\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}\right]=\frac{\partial\tau_{ij}}{\partial x_j}+b_j \tag{18}$$

Equation (16) (and equivalently equation (18)) is called the Cauchy equation of motion (or the Navier equation). Furthermore, note that since equation (16) has been derived purely as a consequence of conservation of momentum, it is called the ‘conservative form’ of the Cauchy equation. On the other hand, equation (18) is derived by conjugating momentum conservation with mass conservation, it is called the ‘non-conservative form’ of the Cauchy equation. The conservative form is useful for computational methods like finite volume method, where conservative forms of differential equations are integrated to discretize the equation. On the other hand, the non-conservative form is used in analytical studies and is more insightful physically, as upon examination, one can see that the LHS of equation (18) is simply the density times material acceleration of the fluid, i.e. $\rho\frac{Dv_i}{Dt}$.