Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 13
Introduction to Traction Vector and Stress Tensor

Viscous Forces in Reynolds Transport Theorem

Many practical forces an engineer encounters have viscosity as an integral part of the inherent physical effects, and therefore, it is crucial to quantify the contribution of viscous forces in the mathematical description of flows. For the same, we start with the general expression of Reynolds transport theorem for linear momentum,

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}(\vec{v}_r \cdot \vec{\eta}) dA \quad (1)$$

Equation (1) is the general equation that applies to both a stationary control volume and a control volume moving with a constant velocity. Furthermore, it can also be applied to a control volume that is accelerating provided the appropriate correction terms are added to the LHS, i.e. the pseudo forces corresponding to the control volume’s acceleration are included.

In continuum mechanics, there are two types of forces that can apply on an arbitrary parcel of fluid (this applies to other types of materials as well). The first type is the surface forces and the second type is the body forces, i.e. $\vec{F} = \vec{F}_{surface} + \vec{F}_{body}$. For a particular control volume, surface forces are the forces that act on the control surface of the control volume. These forces are typically expressed as force per unit area. On the other hand, the forces that act on the volume of the fluid in a control volume are categorized as body forces. These forces are typically expressed as force per unit volume.

Surfaces forces can be mathematically represented using the traction vector, which is demonstrated in figure 1. In the figure, the traction vector $\vec{T}$ applies on the elemental area marked in yellow on the bigger control surface.

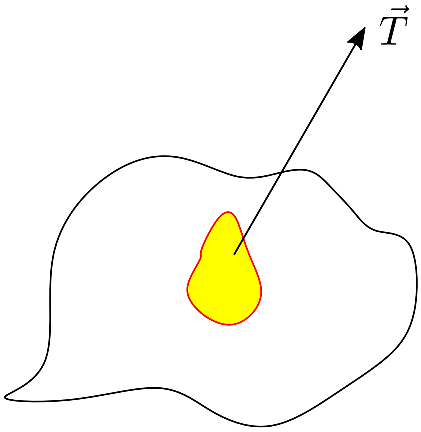

Figure 1: Traction vector on the arbitrary elemental area (yellow region) which is part of the bigger control surface

The traction vector is dependent on the orientation of the surface on which it is being considered, i.e., considering a different elemental surface at a different orientation will give us a different traction vector. Mathematically, the orientation of an elemental surface is represented by the normal vector to it, $\vec{\eta}$, and therefore, to signify the dependence of traction vector on the control surface's orientation, a superscript η is applied to it, i.e. the traction vector is $\vec{T}^{\eta}$, and its components along the three co-ordinates *x, y* and *z* are T_x^{η}, T_y^{η} and T_z^{η} respectively (or equivalently, its components along the three co-ordinates x_1, x_2 and x_3 are T_1^{η}, T_2^{η} and T_3^{η} respectively). On the other hand, we denote the body force per unit volume by $\vec{b}$. Therefore, the LHS of equation (1) simply becomes,

$$\sum \vec{F} = \sum \vec{F}_{surface} + \sum \vec{F}_{body} = \int_{CS} \vec{T}^{\eta} dA + \int_{CV} \vec{b} dV \qquad (2)$$

Next, we seek to express the traction vector $\vec{T}^{\eta}$ in terms of some known parameters, which are the stress tensor components. Note that the stress tensor components are not explicitly known but for most fluids can indeed be expressed as functions of the local velocity gradients in the flow.

To obtain the stress tensor in a fluid, let us consider the arbitrary cuboidal control volume shown in figure 2.

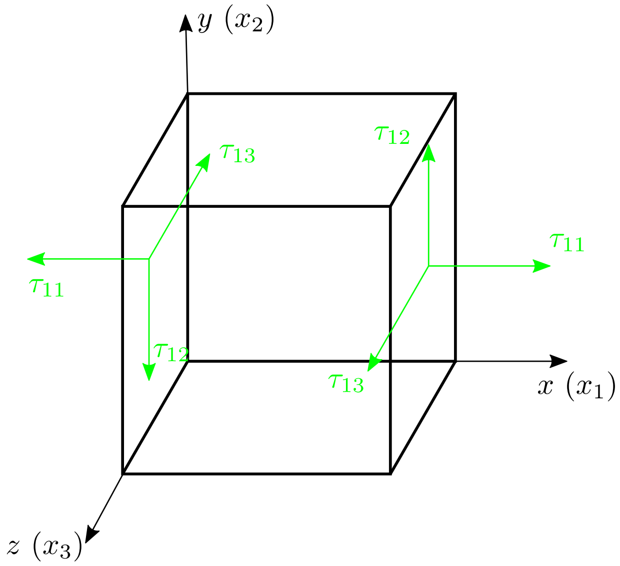

Figure 2: Cuboidal control volume with the stress components on the two faces normal to the *x*-axis (or equivalently x_1-axis) represented

In figure 2, we have chosen the six surfaces to be oriented perpendicular to the co-ordinate axes. For the special case of these surfaces, the traction vector along each surface is represented using the symbol τ rather than T. That is the three components of the traction vector perpendicular to *x* (or x_1), i.e. T_x^x, T_y^x and T_z^x (or T_1^1, T_2^1 and T_3^1) are simply written as τ_{xx}, τ_{xy} and τ_{xz} (or τ_{11}, τ_{12} and τ_{13}) respectively. In the expression for the τ's, we can see that two indices are required – the first index represents the direction normal to the plane on which force is being considered and the second index represents the component of the force. In all, there are 3 x 3 i.e. 9 components for τ_{ij}. Hence, τ_{ij} is more general than a vector, which only has three components. Such variables that require two indices rather than one (and hence having nine components rather than three) are called as tensors (more specifically second order tensors), and τ_{ij} is the stress tensor. It should be noted that a second order tensor maps a vector on to a vector. Similarly, there are higher order tensors – for e.g. fourth order tensors exist, which map a second order tensor onto a second order tensor.

While the stress tensor is specific to the surfaces being perpendicular to the co-ordinate axes, the traction vector doesn't have any such restriction and we should be able to describe a traction vector for any choice of a control surface. Therefore, it is of interest to obtain an expression for an arbitrary tensor $\vec{T}^\eta$ in terms of the stress tensor. Towards this end, let us consider the arbitrary control volume presented in figure 3.

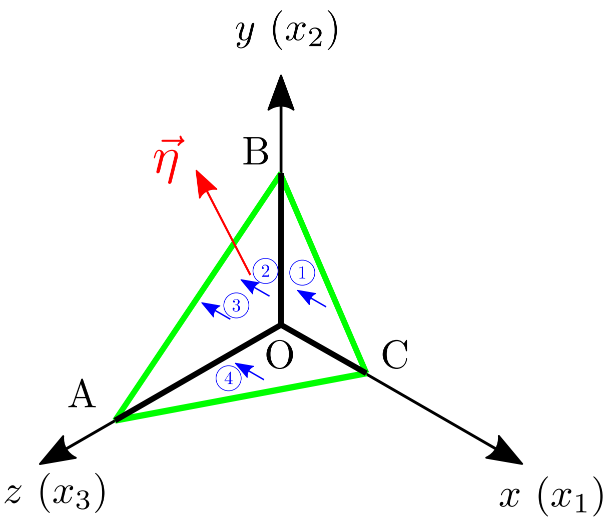

Figure 3: Tetrahedral control volume OABC used to express the traction vector in terms of the stress tensor

In figure 3, we consider the control volume enclosed in the tetrahedron OABC. Let us represent the distance of point O from the surface ABC (i.e. the length of the vertical line dropped from O onto ABC) as *h*. Also, the normal vector to the surface ABC is $\vec{\eta} = \eta_x \vec{e_x} + \eta_y \vec{e_y} + \eta_z \vec{e_z}$ (or $\eta_1 \vec{e_1} + \eta_2 \vec{e_2} + \eta_3 \vec{e_3}$). As can also be seen, the normal to the surface AOB is $-\vec{e_x}$ (or $\vec{e_1}$), the normal to the surface AOC is $-\vec{e_y}$ (or $\vec{e_2}$) and the normal to the surface BOC is $-\vec{e_z}$ (or $\vec{e_3}$) .

We now express the *x*-component force balance for this control volume.

$$m_{CV} a_x = \sum F_{body(x)} + \sum F_{surface(x)} \tag{3}$$

Now, the volume of the control volume is $\frac{1}{3} h A_{\text{ABC}}$, where A_{ABC} is the area of the surface ABC. Further, representing body force per unit volume as $\vec{b}$, equation (3) becomes

$$\frac{1}{3} \rho h A_{\text{ABC}} a_x = F_1 + F_2 + F_3 + F_4 + \frac{1}{3} h A_{\text{ABC}} b_x \tag{4}$$

In equation (4), F_1, F_2, F_3 and F_4 are the forces on the surfaces OBC, OAB, ABC and OAC respectively. The sum of these four forces exhausts the surfaces forces on the control volume OABC. Further, note that the force per unit area on the surfaces OBC, OAB and OAC, as per the definition of stress tensor components, are τ_{zx}, τ_{xx} and τ_{yx} (or τ_{31}, τ_{11} and τ_{21}) respectively. Hence, the *x*-force on the surfaces OBC, OAC and OAC (i.e. F_1, F_2 and F_4 respectively) are these stress tensor components (which are force per unit area) multiplied by the respective surface area, i.e. $-\tau_{zx} A_{\text{OBC}}$, $-\tau_{xx} A_{\text{OAB}}$ and $-\tau_{yx} A_{\text{OAC}}$ (or $-\tau_{31} A_{\text{OBC}}$, $-\tau_{11} A_{\text{OAB}}$ and

$-\tau_{21}A_{\text{OAC}}$) respectively – the minus signs appear because the normal to these surfaces are directed along the negative co-ordinate axes, as specified earlier. Similarly, following the definition of the traction vector, the *x*-force on the surface ABC (i.e. F_3) is $T_x^{\eta}A_{\text{ABC}}$ (or $T_1^{\eta}A_{\text{ABC}}$). Substituting these in equation (4) and doing some re-arranging of terms gives us

$$
\begin{aligned}
&-\frac{1}{3}hA_{\text{ABC}}(\rho a_x - b_x) = \tau_{xx}A_{\text{OAB}} + \tau_{yx}A_{\text{OAC}} + \tau_{zx}A_{\text{OBC}} - T_x^{\eta}A_{\text{ABC}} \\
&(\text{or } -\frac{1}{3}hA_{\text{ABC}}(\rho a_1 - b_1) = \tau_{11}A_{\text{OAB}} + \tau_{21}A_{\text{OAC}} + \tau_{31}A_{\text{OBC}} - T_1^{\eta}A_{\text{ABC}})
\end{aligned} \tag{5}
$$

Now, looking at the surfaces in figure 3, we can recognize that OBC, OAB and OAC are simply the projections of the surface ABC on the *xy*, *yz* and *xz* (or x_1x_2, x_1x_2 and x_1x_2). As a result, the areas of OBC, OAB and OAC are simply the area of ABC multiplied with η_z, η_x and η_y (or η_3, η_2 and η_1) respectively. Hence, substituting these expressions in equation (5), we have

$$
\begin{aligned}
&-\frac{1}{3}hA_{\text{ABC}}(\rho a_x - b_x) = \tau_{xx}A_{\text{ABC}}\eta_x + \tau_{yx}A_{\text{ABC}}\eta_y + \tau_{zx}A_{\text{ABC}}\eta_z - T_x^{\eta}A_{\text{ABC}} \\
&(\text{or } -\frac{1}{3}hA_{\text{ABC}}(\rho a_1 - b_1) = \tau_{11}A_{\text{ABC}}\eta_1 + \tau_{21}A_{\text{ABC}}\eta_2 + \tau_{31}A_{\text{ABC}}\eta_3 - T_1^{\eta}A_{\text{ABC}})
\end{aligned} \tag{6}
$$

Lastly, we take the limit of $h \to 0$, due the which the LHS of equation (6) vanishes. Hence, we have,

$$
\begin{aligned}
&0 = A_{\text{ABC}}(\tau_{xx}\eta_x + \tau_{yx}\eta_y + \tau_{zx}\eta_z - T_x^{\eta}) \Rightarrow T_x^{\eta} = \tau_{xx}\eta_x + \tau_{yx}\eta_y + \tau_{zx}\eta_z \\
&(\text{or } A_{\text{ABC}}(\tau_{11}\eta_1 + \tau_{21}\eta_2 + \tau_{31}\eta_3 - T_1^{\eta})) \Rightarrow T_1^{\eta} = \tau_{11}\eta_1 + \tau_{21}\eta_2 + \tau_{31}\eta_3
\end{aligned} \tag{7}
$$

Following similar procedure, the expressions for the other two components of the traction vector are obtained. Collectively, these three expressions are as presented below.

$$
\begin{aligned}
T_x^{\eta} &= \tau_{xx}\eta_x + \tau_{yx}\eta_y + \tau_{zx}\eta_z \\
T_y^{\eta} &= \tau_{xy}\eta_x + \tau_{yy}\eta_y + \tau_{zy}\eta_z \\
T_z^{\eta} &= \tau_{xz}\eta_x + \tau_{yz}\eta_y + \tau_{zz}\eta_z \\
&\text{or} \\
T_1^{\eta} &= \tau_{11}\eta_1 + \tau_{21}\eta_2 + \tau_{31}\eta_3 \\
T_2^{\eta} &= \tau_{12}\eta_1 + \tau_{22}\eta_2 + \tau_{32}\eta_3 \\
T_3^{\eta} &= \tau_{13}\eta_1 + \tau_{23}\eta_2 + \tau_{33}\eta_3
\end{aligned} \tag{8}
$$

These expressions can be represented in array-matrix form as well –

$$\vec{T} = \underline{\underline{S}}\vec{\eta}$$

where

$$\vec{T} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}, \quad \underline{\underline{S}} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}, \quad \vec{\eta} = \begin{bmatrix} \eta_x \\ \eta_y \\ \eta_z \end{bmatrix} \tag{9}$$

or

$$\vec{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}, \quad \underline{\underline{S}} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}, \quad \vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$