**Advanced Concepts In Fluid Mechanics**
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**Lecture - 12**
**Reynolds Transport Theorem: Angular Momentum Conservation**

## I. Angular Momentum Conservation

Angular momentum is nothing but the moment of linear momentum. Therefore, the rate of change of angular momentum for the system can be written as $\frac{d}{dt}\int_{sys}\vec{r}\times dm\vec{V}$. We seek the R.T.T. equation for angular momentum conservation, i.e. we seek to write this rate of change of angular momentum of the system in terms of the corresponding terms for the control volume. To do so, we set $N=\int\vec{r}\times dm\vec{V}\Rightarrow n=\vec{r}\times\vec{V}$. Therefore, the R.T.T. equation for angular momentum is obtained as,

$$\frac{d}{dt}\int_{sys}\vec{r}\times dm\vec{V}=\frac{\partial}{\partial t}\int_{C}\rho\vec{r}\times\vec{V}dV+\int_{CS}\rho(\vec{r}\times\vec{V})(\vec{V}_r\cdot\hat{\eta})dA. \tag{1}$$

This is essentially Newton's second law for angular motion. It is cautionary that this equation (1) is applicable for non-accelerating control volume, and similar to linear momentum conservation, the angular momentum conservation R.T.T. equation for an arbitrarily moving C.V. will have a correction term.

**<u>Illustrative Example:</u>**

To better understand the conservation of angular momentum, we take the example of a lawn sprinkler. Consider the shown in Figure 1.

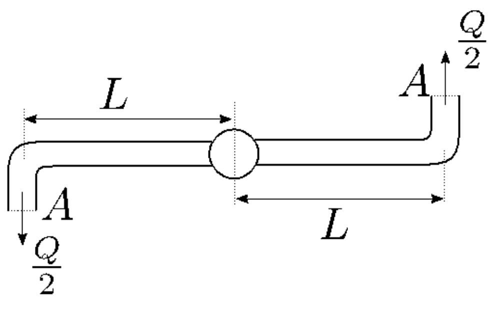


Figure 1: Freely Rotating Sprinkler

We consider a flowrate of $Q$ from this sprinkler, that gets evenly split between the two arms. Consider the area of each of the arms as well as the 'sprinkling' outlets to be $A$. Furthermore, assume that the bent section and the circular central connector are small in dimension compared to the arms, whose length is $L$ each. The sprinkler is freely-rotating, i.e. there is no frictional torque present. The sprinkler arm is bent at the end only to create a rotational effect by changing the direction of flow from radial to tangential.

We analyse this system by two methods.

Method 1: Stationary C.V.

In this method, we consider a stationary control volume that covers a half of the sprinkler, since the system is symmetric. The C.V. is illustrated in Figure 2.

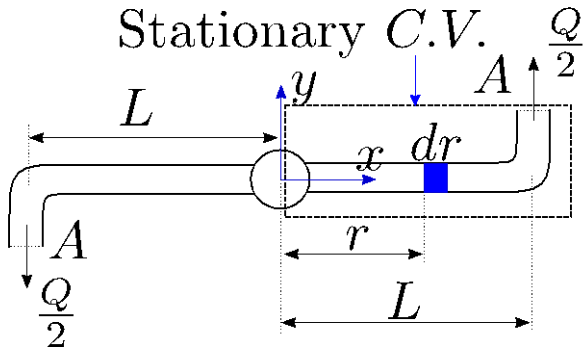


Figure 2: Stationary Control Volume

The angular momentum conservation equation for this C.V. will be,

$$\sum \vec{M} = \frac{\partial}{\partial t}\int_{CV} \rho r\hat{i} \times \left[\vec{\omega}\times\vec{r} + V_r\hat{i}\right] A dr + \rho\left[\{L\hat{i}+0\hat{j}\}\times\{\vec{\omega}\times\{L\hat{i}+0\hat{j}\}+\mathrm{V}_r\ \hat{j}\}\right]\frac{Q}{2}. \qquad (2)$$

In equation (2), the LHS is the sum of all moments acting on the C.V. and the RHS is simply the rate of increase of the systems angular momentum. The system's angular momentum is the integral of angular momentum for all the elementary components. If we consider the angular velocity of the sprinkler to be $\vec{\omega}$, then the expression for the instantaneous angular momentum of the fluid element of length $dr$ at distance $r$ (as presented in Figure 2) from the central connector is $\left(dm\ \vec{r}\times\vec{v}\right)_{\text{element}} = \rho A dr\, r\hat{i}\times\left(\vec{\omega}\times\vec{r}+V_r\hat{i}\right)$, where $V_r$ is the velocity of the fluid element relative the sprinkler arm, and will be in the horizontally rightward direction, i.e. along $\hat{i}$ for most of the C.V. except for a small part near the outlet, where it will be vertically upward, i.e. along $\hat{j}$. Furthermore, the magnitude for $\mathrm{V}_r$ is simply the flow-rate through an arm divided by area of the arm, i.e. $\frac{Q}{2A}$. This expression appears as the integrand in the first term in RHS of equation (2). The second term is the flux of angular momentum out of control surfaces. This corresponds to the second term of equation (1), and with the control surface which is small being the outlet of the sprinkler, we have assumed the integrand to be constant over the control surface. Therefore, the term $\rho(\vec{r}\times\vec{V})$ comes out of the integral and the integral $\int_{CS}(\vec{V}_r\cdot\hat{\eta})dA$ equals $\frac{Q}{2}$. Furthermore, due to the smallness of the bent part of the arm, the position of the outlet from the central connector is approximated as $\{L\hat{i}+0\hat{j}\} = L\hat{i}$.

Since the sprinkler is freely rotating, the moment on it is zero, i.e. the LHS of equation (2) is zero. Further, since $\vec{r} = r\hat{i}$ and $\vec{\omega} = \omega\hat{k}$ (we assume positive angular velocity to be counter-clockwise) equation (2) becomes,

$$\vec{0} = \frac{\partial}{\partial t}\int_0^L \rho r\hat{i} \times \left[\omega r\hat{j} + V_r\hat{i}\right] A dr + \rho\left[L\hat{i} \times (\omega L\hat{j} + \text{V}_r\ \hat{j})\right]\frac{Q}{2}$$

$$\Rightarrow \vec{0} = \frac{\partial}{\partial t}\int_0^L \rho r^2 \omega A dr\hat{k} + \rho L\left(\omega L + V_r\right)\frac{Q}{2}\hat{k} = \frac{\partial}{\partial t}\int_0^L \rho r^2 \omega A dr\hat{k} + \rho L\left(\omega L + \frac{Q}{2A}\right)\frac{Q}{2}\hat{k} \qquad (3)$$

$$\Rightarrow 0 = \rho A\frac{L^3}{3}\frac{d\omega}{dt} + \rho L^2\frac{Q}{2}\omega + \rho L\frac{Q^2}{4A}.$$

Equation (3) is of the form $\frac{d\omega}{dt} + P\omega = Q$, which can be solved taking the use of integrating factor $e^{\int Pdt}$, which is multiplied to the complete equation to aid with the analysis. On the other hand, the steady state angular velocity of the sprinkler can be readily obtained from equation (3) by setting $\frac{d\omega}{dt}$ to zero, giving us $\omega_{\text{steady}} = -\frac{Q}{2AL}$, the negative sign indicating that the sprinkler will actually rotate clockwise.

<u>Method 2: Accelerating C.V. (C.V. rotating with the sprinkler)</u>

The visual representation of this C.V. that is attached to the rotating sprinkler is same as the presented in Figure 2, with the exception that while the fixed C.V. coincided with the sprinkler at the instantaneous moment we did our analysis, this rotating C.V. moves with the sprinkler and coincides with it at all times.

The equation of angular momentum conservation in this rotating C.V. will have a correction term similar to the one we encountered for linear momentum.

The equation for R.T.T. for angular momentum conservation in an arbitrarily moving reference frame is,

$$\sum \vec{M} - \int_{sys} \vec{r} \times dm\vec{a}_{rel} = \frac{d}{dt}\int_{sys} \vec{r} \times dm\vec{V}_{xyz} = \int_{sys} \vec{r} \times dm\vec{a}_{xyz}$$

$$\Rightarrow \sum \vec{M} - \int_{CV} \vec{r} \times dm\vec{a}_{rel} = \int_{sys} \vec{r} \times dm[\vec{a}_{XYZ} - \vec{a}_{rel}], \qquad (4)$$

where, $\int_{CV} \vec{r} \times dm\vec{a}_{rel}$ is the correction term.

We now obtain the correction term. We now substitute the appropriate expressions and continue with equation (4),

$$\sum \vec{M} - \int_{CV} \vec{r} \times \rho A dr\left[\vec{a}_{cv} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + \frac{\partial \vec{\omega}}{\partial t} \times \vec{r} + 2\vec{\omega} \times \vec{v}_r\right] = \int_{sys} \rho\vec{r} \times \vec{v}_{xyz}\left(\vec{v} \cdot \hat{\eta}\right) dA. \quad (5)$$

In writing equation (5), we have made the substitution $dm = \rho A dr$ and the expression for $\vec{a}_{rel}$ is substituted using the derived expression as presented in equation (9) of the last lecture notes. The unsteady term, i.e. $\frac{d}{dt}\int_{sys} \vec{r} \times dm\vec{V}_{xyz}$, is zero because the C.V. and the water are rotating simultaneously and resultantly, with respect to the C.V., there is no net flow. And the term $\vec{v} \cdot \hat{\eta}$ is simply the fluid coming out with respect to the rotating C.V., whose expression is $\rho \frac{Q}{2} L\hat{i} \times \frac{Q}{2A} \hat{j}$, where $\frac{Q}{2A} \hat{j}$ is the velocity at which the water is coming out relative to the sprinkler. Thus, equation (5) becomes,

$$\sum \vec{M} - \int_{CV} \vec{r} \times \rho A dr \left[ \vec{a}_{cv} + \vec{\omega} \times \left( \vec{\omega} \times \vec{r} \right) + \frac{\partial \vec{\omega}}{\partial t} \times \vec{r} + 2\vec{\omega} \times \vec{v}_r \right] = \frac{\rho L Q^2}{4A} \hat{k} \,. \qquad (6)$$

Now, since there is no moment with respect to the rotating C.V., $\sum \vec{M}$ is zero. Since there is no linear acceleration of the C.V., $\vec{a}_{cv}$. Substituting the appropriate expressions for the rest of terms, equation (6) becomes,

$$\begin{aligned}
&-\int_{CV} r\hat{i} \times \rho A dr \left[ \omega\hat{k} \times \left( \omega\hat{k} \times r\hat{i} \right) + \frac{\partial \omega}{\partial t} \hat{k} \times r\hat{i} + 2\omega\hat{k} \times v_r \hat{i} \right] = \frac{\rho L Q^2}{4A} \hat{k} \Rightarrow \\
&-\int_{CV} r\hat{i} \times \rho A dr \left[ -\omega^2 r\hat{i} + \frac{\partial \omega}{\partial t} r\hat{j} + 2\omega v_r \hat{j} \right] = \frac{\rho L Q^2}{4A} \hat{k} \Rightarrow \\
&-\int_0^L \rho A r dr \frac{\partial \omega}{\partial t} r\hat{k} - 2\omega A v_r \rho \int_0^L r dr \hat{k} = \frac{\rho L Q^2}{4A} \hat{k} \Rightarrow \\
&-\rho A \frac{\partial \omega}{\partial t} \frac{L^3}{3} - 2\omega A \frac{Q}{2A} \rho \frac{L^2}{2} = \frac{\rho L Q^2}{4A}
\end{aligned} \qquad (7)$$

In the analysis above, we have made the substitution $v_r = \frac{Q}{2A}$. The final form of equation (7) matches the final form of equation (3), indicating that we have correctly implemented the alternative method (method 2) of C.V. rotating with the sprinkler.

Although this example involves tedious algebra, the purpose of this example is not to obtain the final solution but to demonstrate the two approaches of solving, i.e. one involving fixed C.V. and another involving rotating C.V. attached to the rotating object, and the contrast between them.

Summarily, we have discussed the control volume analysis for mass conservation, linear momentum conservation and angular momentum conservation. In the next lecture, we will start with the dynamics of viscous flows.