Advanced Concepts In Fluid Mechanics
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Lecture - 11
Reynolds Transport Theorem: Arbitrarily Moving Control Volume

I. Arbitrarily Moving C.V.

In some engineering problems, it can be more fruitful to consider a moving reference frame rather than a stationary frame. However, in order to do an analysis in such a reference frame, the governing equations might have to be modified to be applicable in this new reference frame. Therefore, in this lecture, we will first consider an arbitrarily moving reference frame and assess how to make any modifications to the governing equations that would be required.

a. Chasle's Theorem

Let us consider a vector $\vec{A}(t)$ which is fixed in a rotating reference frame rotating with angular velocity ω in the counter-clockwise direction. However, $\vec{A}$ will move in a fixed reference frame as presented in figure 1.

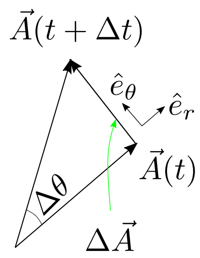

Figure 1: Motion in a fixed reference frame of vector $\vec{A}(t)$ that is fixed in a c.c.w. rotating reference frame with angular velocity of ω.

The expression for $\frac{d\vec{A}}{dt}$ in the fixed reference frame is given as,

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta A\hat{e}_\theta}{\Delta t} = \lim_{\Delta t\to 0}\frac{A\Delta\theta\hat{e}_\theta}{\Delta t} = A\omega\hat{e}_\theta. \quad (1)$$

In equation (1), we have made use of the facts:

(a) the $\hat{e}_r - \hat{e}_\theta$ system is oriented such that $\hat{e}_r$ is along $\vec{A}(t)$ and $\hat{e}_\theta$ is perpendicular to it,
(b) for small enough Δt, $\Delta\theta$ will be so small that we can consider $\tan\Delta\theta \approx \Delta\theta$ and the angle between $\vec{A}(t)$ and $\Delta\vec{A}$ is 90^0, due to which, $\Delta A = \vec{A}(t)\tan(\Delta\theta) \approx A\Delta\theta$.

Now, angular velocity of the rotating reference frame can be expressed as $\vec{\omega} = \omega \hat{e}_z$. Taking the cross-product of this angular velocity with $\vec{A}$ is,

$$\vec{\omega} \times \vec{A} = \omega \hat{e}_z \times A\hat{e}_r = \omega A \hat{e}_\theta. \tag{2}$$

Substituting this expression in equation (1), we write,

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}. \tag{3}$$

While equation (3) is the expression for the time-derivative of a vector $\vec{A}$ that is fixed in the rotating frame, this is also the additional term for the transformation of time-derivative from a fixed to a rotating reference for any vector, even if moving with respect to the rotating reference frame. Summarily, if we represent a fixed reference frame by subscript XYZ and an arbitrarily moving reference frame by subscript xyz, then the counterpart of equation (3) for any arbitrary vector $\vec{B}$, which is allowed to move w.r.t. xyz also, is,

$$\left.\frac{d\vec{B}}{dt}\right|_{XYZ} = \left.\frac{d\vec{B}}{dt}\right|_{xyz} + \vec{\omega} \times \vec{B}. \tag{4}$$

Equation (4) is called Chasle's theorem.

b. Fluid Acceleration in Arbitrarily Moving Frame

When considering conservation of fluid momentum, we are primarily concerned with fluid acceleration, which essentially implies a second derivative of position vector with time. Therefore, we need to find an expression relating the second derivative of position with time in the fixed reference frame and the arbitrarily moving reference frame. If the origin of the moving reference frame xyz is located at $\vec{R}_0$ with respect to the fixed reference frame XYZ, then the position vector w.r.t. the fixed reference frame, denoted by $\vec{R}$ is expressed in terms of the position vector w.r.t. the moving reference frame, denoted by $\vec{r}$, and the vector $\vec{R}_0$ as $\vec{R} = \vec{r} + \vec{R}_0$. Taking a time-derivative, we have,

$$\left.\frac{d\vec{R}}{dt}\right|_{XYZ} = \left.\frac{d\vec{R}_0}{dt}\right|_{XYZ} + \left.\frac{d\vec{r}}{dt}\right|_{XYZ}. \tag{5}$$

Substituting $\left.\frac{d\vec{R}}{dt}\right|_{XYZ}$ as $\vec{B}$ in equation (4), we get,

$$\left.\frac{d^2\vec{R}}{dt^2}\right|_{XYZ} = \left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ} + \frac{d}{dt}\left(\left.\frac{d\vec{r}}{dt}\right|_{XYZ}\right)_{XYZ}. \tag{6}$$

Now, using equation (4) on the second term on RHS in equation (6), we proceed as,

$$\left.\frac{d^2\vec{R}}{dt^2}\right|_{XYZ} = \left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ} + \frac{d}{dt}\left[\left.\frac{d\vec{r}}{dt}\right|_{XYZ}\right]_{xyz} + \vec{\omega}\times\left.\frac{d\vec{r}}{dt}\right|_{XYZ}$$

$$= \left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ} + \frac{d}{dt}\left[\left.\frac{d\vec{r}}{dt}\right|_{xyz} + \vec{\omega}\times\vec{r}\right]_{xyz} + \vec{\omega}\times\left(\left.\frac{d\vec{r}}{dt}\right|_{xyz} + \vec{\omega}\times\vec{r}\right) \quad (7)$$

$$= \left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ} + \left.\frac{d^2\vec{r}}{dt^2}\right|_{xyz} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} + \vec{\omega}\times\frac{d\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Put together, equation (7) gives,

$$\left.\frac{d^2\vec{R}}{dt^2}\right|_{XYZ} = \left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ} + \ddot{\vec{r}}_{xyz} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{xyz}) + \dot{\vec{\omega}}\times\vec{r}_{xyz} + 2\vec{\omega}\times\vec{V}_{xyz}. \quad (8)$$

Interchanging the first and second terms of RHS of equation (8) and re-writing, we have,

$$\vec{a}_{XYZ} = \vec{a}_{xyz} + \vec{a}_{CV} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{xyz}) + \dot{\vec{\omega}}\times\vec{r}_{xyz} + 2\vec{\omega}\times\vec{V}_{xyz} = \vec{a}_{xyz} + \vec{a}_{rel}. \quad (9)$$

where, $\vec{a}_{XYZ}$ is $\left.\frac{d^2\vec{R}}{dt^2}\right|_{XYZ}$, the acceleration of fluid particle w.r.t. the fixed reference frame, $\vec{a}_{xyz}$ is $\ddot{\vec{r}}_{xyz}$, the acceleration of fluid particle w.r.t. the moving reference frame, and $\vec{a}_{CV}$ is $\left.\frac{d^2\vec{R}_0}{dt^2}\right|_{XYZ}$, the acceleration of the moving frame itself.

In RHS of equation (9), the first term $\vec{a}_{xyz}$ is the acceleration of the fluid particle with respect to the moving frame and the rest of the terms are collectively termed as the relative acceleration $\vec{a}_{rel}$.

c. Reynold's Transport Theorem in Arbitrarily Moving Frame

Having obtained the expression for acceleration of a fluid particle w.r.t. an arbitrarily moving reference frame, equation (9), we write the Reynolds Transport Theorem (R.T.T.) for arbitrarily moving control volume.

Starting with the general expression for Reynold's theorem,

$$\frac{d}{dt}\int_{sys} dm\vec{V}_{xyz} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}_{xyz}dV + \int_{CS} \rho\vec{V}_{xyz}(\vec{V}_r\cdot\hat{\eta})dA, \quad (10)$$

we recognize that the relative velocity of R.T.T. $\vec{V}_r$ is with respect to the moving frame xyz, and, the the L.H.S. can be simplified to,

$$\frac{d}{dt}\int_{sys} dm\vec{V}_{xyz} = \int_{sys} dm\vec{a}_{xyz} = \int_{sys} dm\vec{a}_{XYZ} - \int_{sys} dm\vec{a}_{rel}. \quad (11)$$

In equation (11), we have substituted the expression for $\vec{a}_{xyz}$ from equation (9). Since $\vec{a}_{XYZ}$ is the acceleration w.r.t. a fixed reference frame, it equals the force per mass in accordance with the second law of motion, i.e.

$$\int_{sys} dm\vec{a}_{XYZ} = \sum \vec{F}_{sys}. \qquad (12)$$

Since R.T.T. was derived for vanishingly small time interval Δt, the C.V. coincides with the system and $\sum \vec{F}_{sys} = \sum \vec{F}_{cv}$. Therefore, equation (10) for an arbitrarily moving control volume is obtained as,

$$\sum \vec{F}_{cv} - \int_{CV} dm\vec{a}_{rel} = \frac{d}{dt}\int_{sys} dm\vec{V}_{xyz} = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}_{xyz} dV + \int_{CS} \rho\vec{V}_{xyz}(\vec{V}_r \cdot \hat{\eta})dA, \qquad (13)$$

which is the expression for R.T.T. in an arbitrarily moving control volume.

Summarily, we can use the Reynolds transport theorem for an arbitrarily moving control volume for momentum conservation, but with the correction term that is the second term in LHS of equation (13). This term is an additional effective force, i.e. a pseudo force that acts on a fluid element when observing it from an arbitrarily accelerating reference frame.

So, this pseudo force is a combination of four different components, corresponding to the four terms of $\vec{a}_{rel}$. The first component is the classical pseudo force we came across when studying dynamics of point particles in accelerating reference frames. The second component is the centrifugal force due to rotation of the frame, another classical force previously studied in the context of point-particle dynamics. Centrifugal force in a rotating from is the counterpart of centripetal acceleration in an inertial frame (fixed frame is a subset of inertial frame). While the former appears as a subtraction to the force term in the force balance in the rotating frame, the latter appears as an addition to the acceleration in force balance. Therefore, care must be taken not to double-count this term by taking both centrifugal force as well as centripetal acceleration in a force balance equation. The third component is called Euler Force, which is the force due to angular acceleration of the reference frame. The fourth and final component is called as the Coriolis force. This is a lateral force that acts on a fluid particle due to its translation relative to a rotating reference frame. This force contributes significantly to ocean currents with the earth being the rotating frame. Hence, Coriolis force is very important in fluid mechanics and in the broad purview of geophysical fluid dynamics.

Illustrative Example:

To elucidate the concept presented in this lecture, we consider the illustrative example of the motion of a rocket. Consider the rocket going vertically upward with speed V_R, and ejecting burnt fuel vertically downward at speed V_e (relative to rocket), out of the small opening of area A_e. The system schematic is presented in Figure 2. Assume gravity acts vertically downwards. Initial mass of rocket with the fuel was m_0 and ρ is the density of the fuel. Obtain an expression for V_R.

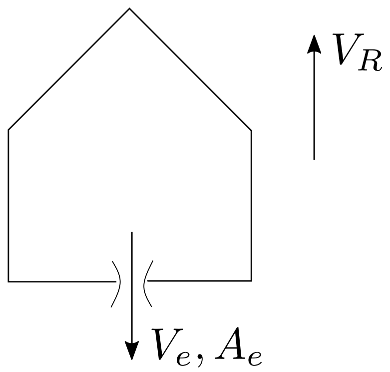

Figure 2: Schematic of the Illustrative Problem

<u>Soln. –</u>

To solve this problem, consider the C.V. co-incident with the rocket, presented with Figure 3. The forces on the rocket will be gravity (acceleration due to gravity is g) and a vertically downward drag force F_D, as illustrated in Figure 3 as well.

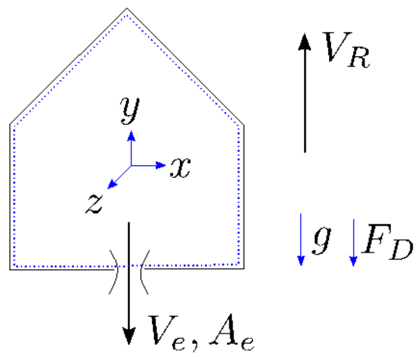

Figure 3: C.V. to solve the Illustrative Problem

Hence, writing the y-component of momentum balance equation for this control volume,

$$-F_D - M_R g - M_R a_{rel(y)} = \frac{d}{dt}\int_{sys} dm V_{y(xyz)} = \frac{\partial}{\partial t}\int_{CV} \rho V_{y(xyz)} dV + \int_{CS} \rho V_{y(xyz)} (\vec{V}_r \cdot \hat{\eta})\ dA \tag{14}$$
$$\Rightarrow -F_D - M_R g - M_R a_{CV(y)} = -\rho A_e V_e^2,$$

where M_R is the instantaneous mass of the rocket. In equation (14), In writing the bottom equation in equation (14) from first, note that:

(a) The frame is accelerating but not rotating and therefore, $a_{rel(y)}$ has only the first component and is thus equal to $a_{CV(y)}$.

(b) Velocity of fuel in the rocket has zero velocity w.r.t. the rocket (and the attached C.V.), i.e. $V_{y(xyz)} = 0$.

(c) The second term in RHS of top equation in equation (14) applies only at the burnt fuel outflow hole and equals $-\rho A_e V_e^{\,2}$.

$$
\begin{aligned}
&\left.\frac{dm}{dt}\right|_{sys} = \left.\frac{dm}{dt}\right|_{sys} + \int_{CS} \rho(\vec{V}_r \cdot \hat{\eta})dA \\
&\Rightarrow 0 = \left.\frac{dm}{dt}\right|_{sys} + \dot{m}_e v_e = \left.\frac{dm}{dt}\right|_{sys} + \rho A_e v_e.
\end{aligned} \qquad (15)
$$

Integrating equation (15), the expression for M_R is,

$$
M_R = m_0 - \rho A_e v_e t. \qquad (16)
$$

The expression for $a_{CV(y)}$ is simply the acceleration of rocket, i.e. $\frac{dV_R}{dt}$. Substituting this, and M_R from equation (16) in equation (14), and neglecting the drag force F_D, we have,

$$
\frac{dV_R}{dt} = \frac{\rho A_e V_e^2}{m_0 - \rho A_e V_e t} - \text{g}. \qquad (17)
$$

Integrating equation (17) will give the expression for V_R as a function of time.

We discuss the roles of different forces. First, the role of the drag force and of gravity is to push the rocket vertically down. In contrast, the role of the momentum flux is to proper the rocket upwards. In other words, it can be said that the rocket goes up by either Newton's third law or the conservation of momentum.

Before concluding the lecture, we discuss one conceptual anomaly needs to be resolved. While deriving the R.T.T. for the arbitrarily moving CV, equation (13), we had considered a fixed mass, and therefore, $\frac{d}{dt}\int_{sys} dm\vec{V}_{xyz} = \int_{sys} dm\vec{a}_{xyz}$ held true. However, in the illustrative example above, we encounter a variable mass. Hence, we could be required to incorporate the effect of variable system mass when doing our analysis for the illustrative example. However, this issue gets circumvented by 'freezing' or 'snapping' the phenomenon at an instant of time when the system and control volume are merging. That is, the instantaneous mass of the system and the instantaneous mass of the control volume are equal due to convergence of the system and control volume in the limit of $\Delta t \rightarrow 0$.