Advanced Concepts In Fluid Mechanics
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Lecture - 10
Reynolds Transports Theorem: Mass and Linear Momentum Conservation

I. Illustrative Example on Conservation of Mass Integral Formulation

In the previous lecture, we obtained the differential form of the conservation of mass using the Reynolds transport theorem. However, even retained in its integration form, the Reynolds transport theorem used for mass conservation can be useful for certain problems. We illustrate this point with an example.

Consider a flat plate of length L, coming down with a constant vertical velocity V_0 towards a fixed boundary. Between the moving plate and the fixed boundary, there is a fluid with constant density ρ, which is getting squeezed out on both the sides due to the decreasing gap between the plate and fixed boundary. Assume that the velocity profile at the outlet is given by u whose expression is,

$$u = 4u_0\left[\frac{y}{h} - \frac{y^2}{h^2}\right]. \tag{1}$$

where h is the instantaneous gap between the two plates. The complete system is illustrated in Fig. 1.

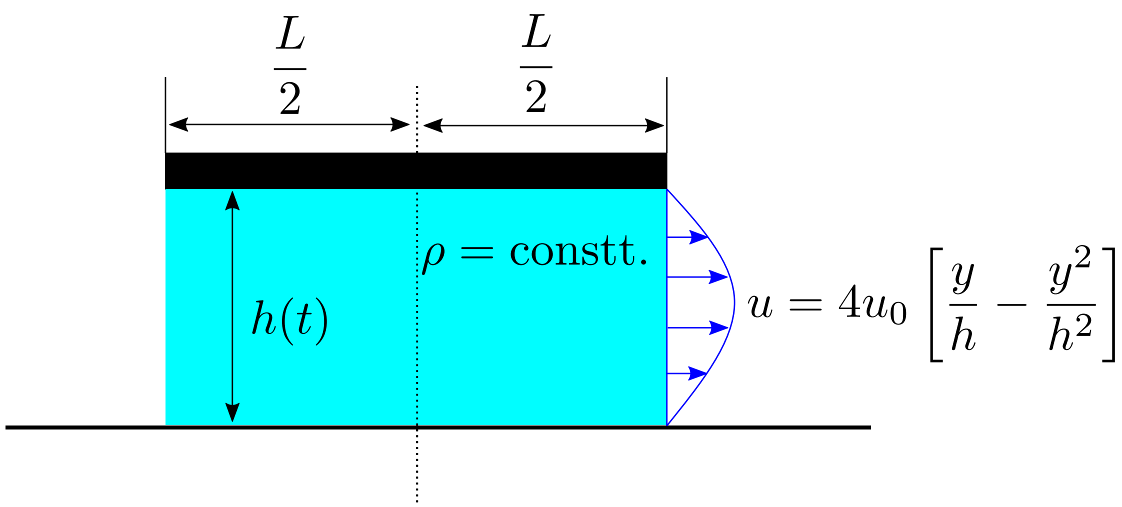

Figure 1: A flat plate approaching a fixed bottom with fluid draining from the sides as per the velocity distribution specified

Find:

(i) the expression for h as a function of time t, given that the starting height of the plate above the bottom fixed boundary is h_0

(ii) the expression for u_0 as a function of time t

Soln. – The answer for the first part is straightforward. The velocity of the top plate being V_0 implies $\frac{dh}{dt}$ is $-V_0$. Integrating from $t=0$ to t, we have,

$$\int_{h_0}^{h(t)} dh = \int_{t=0}^{t} \frac{dh}{dt} dt \Rightarrow h(t) - h_0 = -V_0(t-0) \Rightarrow h(t) = h_0 - V_0 t. \qquad (2)$$

For the second part, we have to consider a control volume. We have two choices for control volume, either a fixed control volume that coincides with the fluid for the time-instant we are considering, or a shrinking control volume with that is coincident with the fluid for all time. Either of these control volumes is taken over the right half only as there is no flow at the centreline due to symmetry. These control volumes are presented in Fig. 2 below. The representation of either control volume is the same, i.e. ABCD, but difference between the fixed and the shrinking control volumes is that the face CD is stationary for the fixed control volume and is moving vertically downward with speed V_0 for the shrinking control volume.

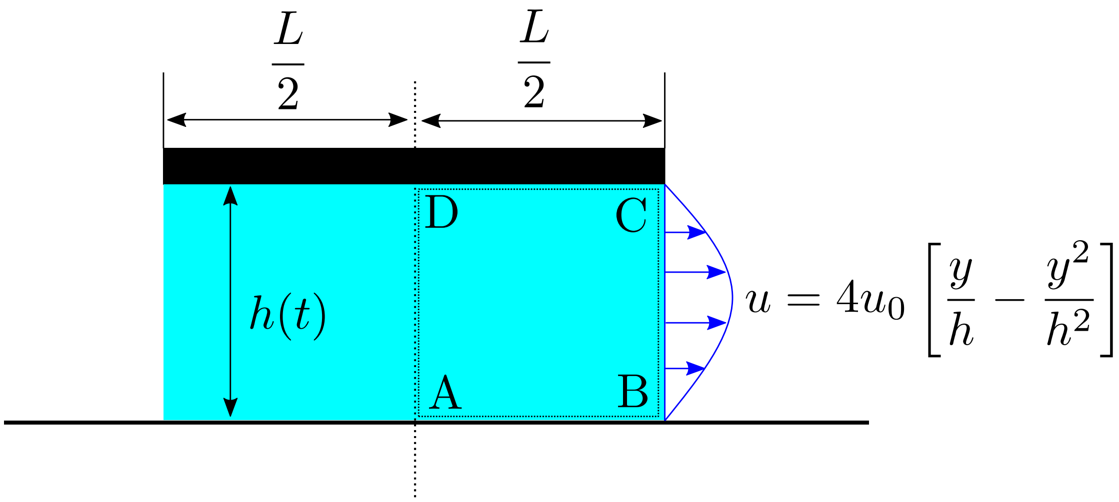

Figure 2: Control Volume (Fixed and Shrinking) represented

We consider the fixed control volume first. The mass conservation equation is,

$$0 = \left.\frac{\partial m}{\partial t}\right|_{CV} + \int_{CS} \rho(\vec{v}\cdot\hat{\eta})dA \qquad (3)$$

Since the control volume is fixed, the first term on RHS of equation (3) is zero, and the third term consists of two components. The first component is due to the fluid entering at the top, and amounts to an mass outflow rate of $-\frac{1}{2}\rho V_0 Lb$, where b is the width of the plate inside the plane of paper. The second component is due to the outflow at the side and amounts to an mass outflow rate of $\int_{y=0}^{y=h} \rho b u(y) dy$. This gives,

$$0 = -\frac{1}{2}\rho V_0 Lb + \int_{y=0}^{y=h} \rho b u(y) dy \Rightarrow \frac{2}{3} b u_0 h = \frac{1}{2} V_0 Lb \Rightarrow u_0 = \frac{3LV_0}{4(h_0 - V_0 t)}. \qquad (4)$$

In equation (4), we have employed the expression for $u(y)$ and the obtained expression for $h(t)$ as given in equation (2).

Considering the shrinking control volume now, the first term of equation (3) will be non-zero, and will be expressed as $\frac{d}{dt}\left(\rho bh\frac{L}{2}\right)=-\frac{\rho bLV_0}{2}$. On the other hand, the second term, i.e. the boundary flux term will constitute not two terms but only the outward flow at the sides, i.e. $\int_{y=0}^{y=h}\rho bu(y)dy$. However, combined together, these will give us equation (4) and thus the same solution as obtained with a fixed control volume.

Summarily, we have observed that while solving with the two different control volumes require distinctive paths, the final solution converges to the same. Also, it should be noted that in terms of pictorial representation, both control volumes look identical, i.e. ABCD in Fig. 2. However, they are distinctive in terms of a moving boundary, which had to be brought out in terms of textual description of the control volume. Therefore, it is crucial that one should properly represent the control volume using a combination of pictorial and textual description when doing a Reynolds transport theorem analysis of a flow problem.

II. Conservation of Linear Momentum

We now obtain the integral form of conservation of linear momentum from the Reynolds transport theorem. To do so, the elementary extensive property dN is $\vec{v}dm$, and the corresponding intensive property n is $\vec{v}$. Substituting these in the Reynolds transport theorem gives,

$$\frac{d}{dt}\int_{\text{system}}\vec{v}dm=\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV+\int_{CS}\rho\vec{v}(\vec{v}_r\cdot\hat{\eta})dA. \tag{5}$$

The LHS of equation (5) is the time rate of change of momentum for the control mass system, and as per Newton's second law, it equals to the force on the control mass system. Since the control mass is co-incident with the control volume at the time-instant when equation (5) is applied, the force on control mass system is equal to the force on the control volume. Therefore, equation (5) can be written as,

$$\sum\vec{F}_{\text{system}}=\sum\vec{F}_{CV}=\frac{d}{dt}\int_{\text{system}}\vec{v}dm=\frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV+\int_{CS}\rho\vec{v}(\vec{v}_r\cdot\hat{\eta})dA. \tag{6}$$

While there isn't a relation between the force and momentum change rate for a control volume from fundamental physical principles, we have obtained a relation between these two in equation (6). This has been possible thanks to the Reynolds transport theorem, which relates the control mass system description to the control volume description, and we have Newton's second law hold true for the former. In the expression for control volume, apart from the expression for time-rate of change of momentum in the control volume (i.e. the first term in RHS of equation (6)), there is a term corresponding to net flux of momentum out of the control volume as well (i.e. the second term in RHS of equation (6)). Equation (6) is sometimes called as momentum theorem, but it is fundamentally Reynolds transport theorem applied for momentum conservation.

We now consider an example to elucidate the conservation of linear momentum. Consider the flow over a flat plate where the fluid velocity is uniformly u_∞ at the start of the plate and is given by the expression $u = u_\infty\left[2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2\right]$ at a length L downstream, where δ is the thickness of the boundary length at length L downstream. Find the force on the plate due to the fluid. The system is represented in figure 3. Consider the density of the fluid to be constant.

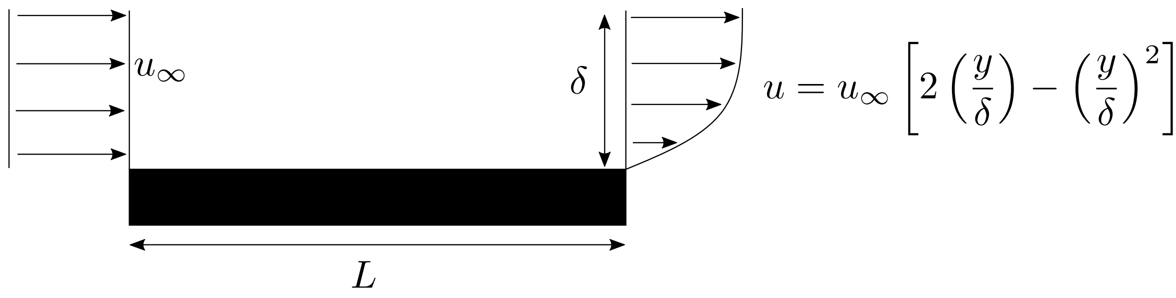

Figure 3: Flow over a plate starting at the left end and developing to a boundary layer of thickness δ at length L downstream.

Soln. – To solve this problem, we first need to choose a control volume. We first consider the simple choice of a rectangular box ABCE. The problem is this control volume is that due to the different velocity profiles at AE and BC, we would have a mass flux at EC, which would complicate the analysis. On the other hand, considering once face of the control volume co-incident with a streamline ensures there is no mass or momentum flux across this face. Therefore, we consider a control volume ABCD, where CD is along a streamline. Both these control volumes are illustrated in Fig. 4.

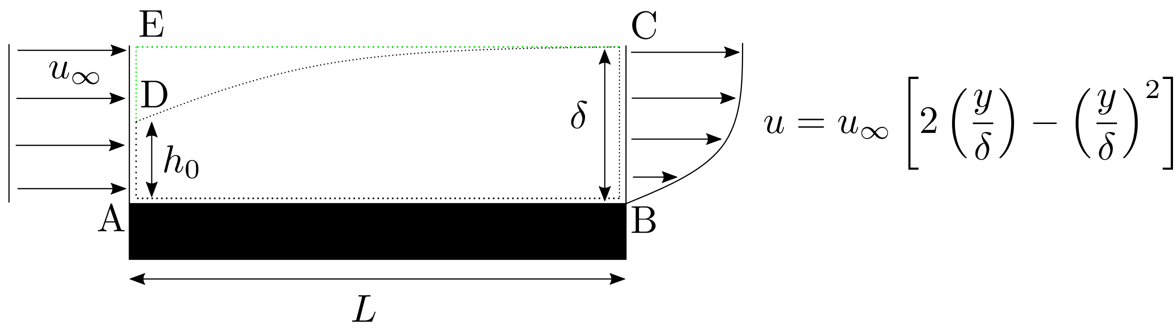

Figure 4: Control volumes ABCD and ABCE for applying Reynolds transport theorem conservation equations.

Therefore, we first use mass conservation to obtain the expression for h_0. Since there is no mass flux across CD due to being co-incident to a streamline, and there is no mass flux across AB since it is co-incident with the plate, the mass entering over AD should equal the mass leaving over BC, i.e.

$$u_\infty h_0 \cdot \rho b = \rho b\int_{y=0}^{\delta} u dy \Rightarrow h_0 = u_\infty \delta \int_{\frac{y}{\delta}=0}^{1}\left(\frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right) \qquad (7)$$

In equation (7), b is the width of the plate.

Now, we will apply this momentum balance. We re-iterate equation (6),

$$\sum \vec{F}_{CV} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}(\vec{v}_r \cdot \hat{\eta}) dA. \tag{8}$$

We consider only the rightward direction component of equation (8), taken as the positive $x-$ direction. If the force per unit width on the plate due to the fluid is taken as F in the rightward direction, then the force per unit width on the fluid due to the plate is F in the leftward direction. Therefore, The LHS of equation (8) becomes $-Fb$. Since the control volume is fixed, the first term in RHS of equation (8) is zero. Further, there is no momentum flux across AB and CD. The momentum flux across DA is $-\rho u_\infty h_0 b = \rho u_\infty^2 \delta b \int_{\frac{y}{\delta}=0}^{1} \left(\frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right)$ and that across BC is $\rho u_\infty^2 \delta b \int_{\frac{y}{\delta}}^{1} \left(\frac{u}{u_\infty}\right)^2 d\left(\frac{y}{\delta}\right)$. Put together and substituted in equation (8), this gives us,

$$-Fb = \rho u_\infty^2 \delta b \int_{\frac{y}{\delta}}^{1} \frac{u}{u_\infty}\left(\frac{u}{u_\infty} - 1\right) d\left(\frac{y}{\delta}\right) \Rightarrow F = \rho u_\infty^2 \delta \int_{\frac{y}{\delta}}^{1} \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right) \tag{9}$$

Substituting the expression for $\frac{u}{u_\infty}$ in equation (9), we get $F = \frac{2\rho u_\infty^2 \delta}{15}$.