Conduction and Convection Heat Transfer
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Lecture - 30
Transient Conduction: Semi - Infinite Slab - I

Let us say that instead of having a given temperature boundary condition we have another problem where we use the given convective heat transfer coefficient boundary condition.

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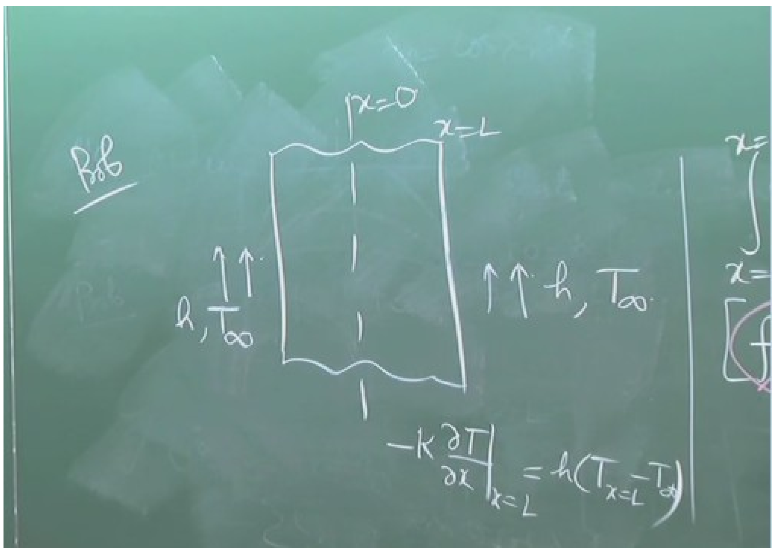

Let us say same infinite slab. Let us say that some fluid is flowing outside the slab with heat transfer coefficient h, and temperature T infinity. So, the only difference from the previous problem is that there is a different boundary condition. Previous problem what type of boundary condition it was? It was a Dirichlet type of boundary condition where the temperature was specified. What type of boundary condition is not there for this problem?

What is the boundary condition here? What is the boundary condition at this surface? So, physically what is the boundary condition? Whatever is the rate at which heat is transferred here at the same rate heat is transferred from here to the surroundings. Let us say that the system is at a greater temperature then the surroundings. So that means minus this is x equal to L, this is x equal to zero, minus K delta T delta x, at x equal to L is equal to h into T at x equal to L minus T infinity.

Now again the confusion always remains is that we are assuming that it is an unsteady state problem still we are assuming that the rate at which heat is transferred is the same rate at which heat get transferred to the surroundings. So is there any kind of inconsistency between this and the consideration of unsteady state. There is no inconsistency because a surface cannot store thermal energy does not matter whether it is steady or unsteady surface cannot store thermal energy.

So, at whatever rate heat is transferred to the surface at that instance at the same rate heat will be transferred from that surface. So, this boundary condition is the only difference between the previous problem and this problem. So, let us workout that let us try to follow the solution of the previous problem and then see what are the changes? So, at x equal zero delta theta delta x here we can also solve half of the domain and at x equal to L.

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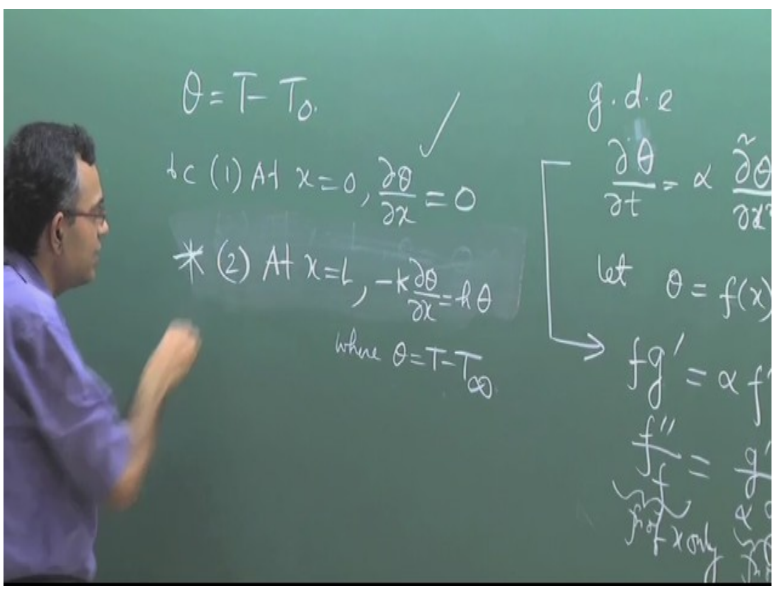

Here, we define as T minus T infinity, right. Minus k delta theta, delta x equal to h into theta, theta is T minus T infinity. This At x equal to l. So, this is the only change and let us quickly see how this change is reflected in the solution. So, this part of the solution remains the same because this is before applying the boundary conditions. So before applying the boundary conditions whatever was the solution general solution is the same.

So ultimately your general solution is when you apply the boundary condition so theta is summation of Cn e to the power minus alpha lambda n square t into cos lambda n x. How did we get this solution we get this solution by substituting this boundary condition, right and this boundary condition does not change from the previous problem to this problem so this form is fine but lambda n is different?

So how do you get lambda n? You have to use the second boundary condition.

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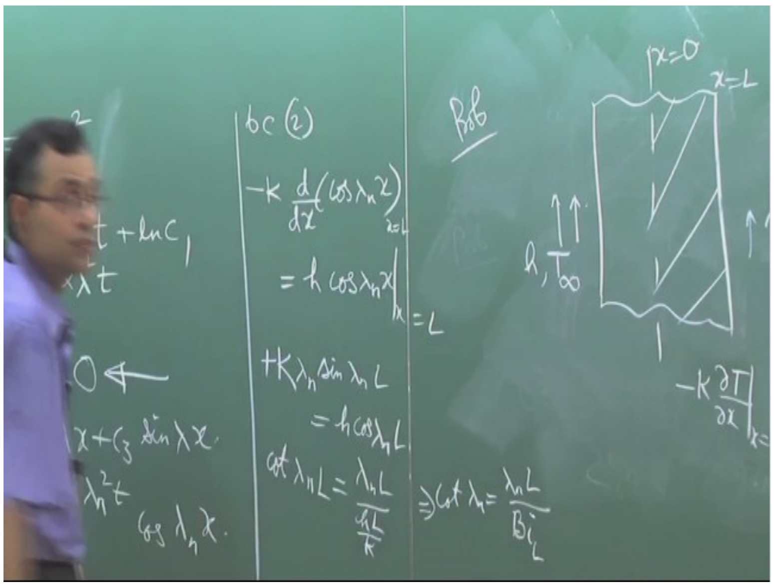

Minus k remember when you are differentiating with respect to x only f is important not g because g is not function of x, right. So minus k this will be basically df and dx, df and dx is d dx of cos lambda nx and this is h into theta, theta will eventually be f, g gets cancelled from both sides. So minus k minus will get cancelled so k lambda n sin lambda n L is equal to h cos lambda n L. So, you can write cot lambda n L equal to lambda n L by hL by K, right.

So, what we have done is we have multiplied by L because we want to use the none dimensional number the Biot number. Biot number is hL by k. So, cot lambda n L is equal to lambda nL by Biot number based on L. So, this is the iL value. See the iL value determination is not very straightforward because this is a transcendental equation. If you want to get a graphical feel of the solution then we can draw it here. So, this is lambda nL.

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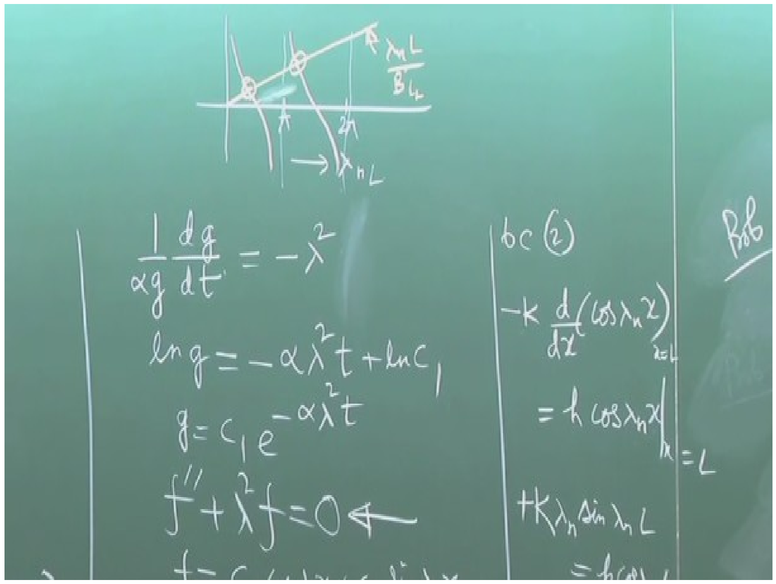

Pi two Pi like that we can sketch. So, cot lambda nL will be something like this, right. In this way, you will have infinite number of such segments and the other part of the solution is this is lambda nL by Biot number. This is y equal to mx, a straight line passing through the origin. So, this point of intersection are the iL values. So, if you are familiar with Met lab what you can do, in Met lab.

You can draw the graph of cot lambda L and you may or say cot x and you may draw the graph of x/ Biot number and find out the points of intersection. Those points of intersections are the Eigen values of this problem. For the previous problem this was like straight forward two in plus one into Pi by two. For this problem, it has to be obtained from this. Now what about the constant SC?

So, we will quickly see how to get constant Cn because we have got lambda n see all these problems are Eigen value problems just like in Linear algebra you require, in Eigen Value problems you require to find out the Eigen Value problems and vector. Similarly, differential equation you require to find out the Eigen Value and Eigen Value function. So, Eigen Value we have already determined now the Eigen functions are related by orthogonality condition.

And we have to derive the orthogonality condition here.

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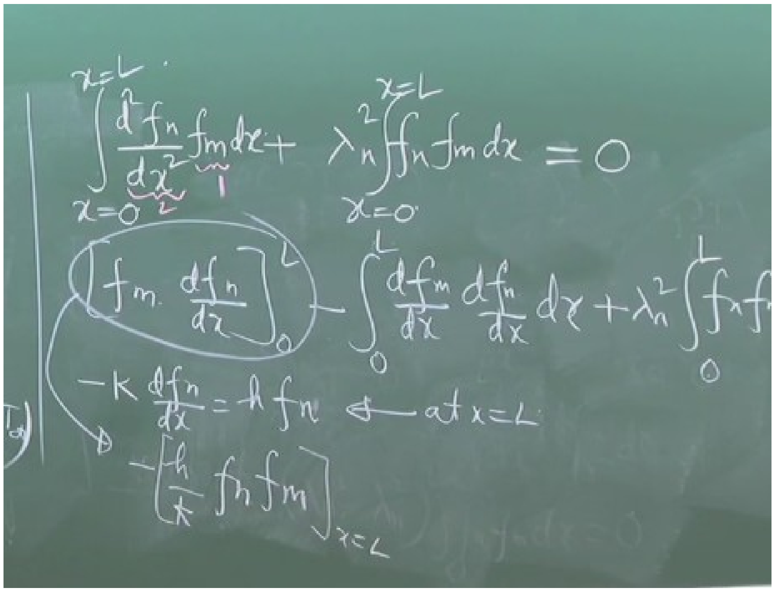

So, we will use this fm into dfn/dx. Why I am separately here? Because here the boundary condition is different so we have to make sure that we derive the orthogonality condition freshly from the boundary condition of this problem. So, what is the boundary condition of this problem at x equal to zero it is the same. So, at x equal to zero this term is zero. But x equal to L this term is not zero, right. At x equal to L what is this term? Minus k dfn/dx equal to age into fn.

This is minus k delta theta/ delta x equal to age into theta. This is at x equal to L. So, what will this become in place of d fn/dx we will have minus h/k fn fm, right. Because at x equal to zero any way df/dx is zero at x equal to L it is not zero but dfn/ dx is related to fn. So, we have written this now what we will do we will swap n and m and subtract and you can see because of the symmetry in fn fm these terms will be cancelled.

So, that is why see eventually sometimes the ignorance is a blessing if you do formula based study then you need not go through it you eventually know orthogonality condition will come some way or the other. So, you eventually write only the orthogonality condition but there may be a tricky boundary condition when orthogonality condition itself is not satisfied and then you cannot use the separation of variable method to solve that problem.

So, for every typical boundary condition you have to look for the orthogonality condition. Here fortunately the orthogonality condition can be obtained because when you swap fn and fm and subtract this term will be zero. So, eventually you will get this. So, although you get the same final answer for the orthogonality condition so far as the orthogonality condition is concerned but the root of getting that orthogonality condition is different for different problems.

Then the remaining solution is very similar to the previous one so you can get Cm from here just lambda will be different from the previous problem. So, only difference with the previous problem is Eigen value otherwise the structure of the problem is very, very similar. So, to summarize what we have discussed today some unsteady state problems where you have problem solved by the method of separation of variables.

So, we have addressed the method of separation of variables with the help of some examples of infinite slabs. We will later on consider some of examples which we call as semi-infinite problems and we will take it up in the next lecture. Thank you very much.

So, we have been discussing about various issues related to unsteady heat conduction and we will discuss about one final issue about unsteady heat conduction before concluding this chapter and that issue is about unsteady heat conduction in a semi-infinite medium. So, let us discuss what it is? So, let us say that

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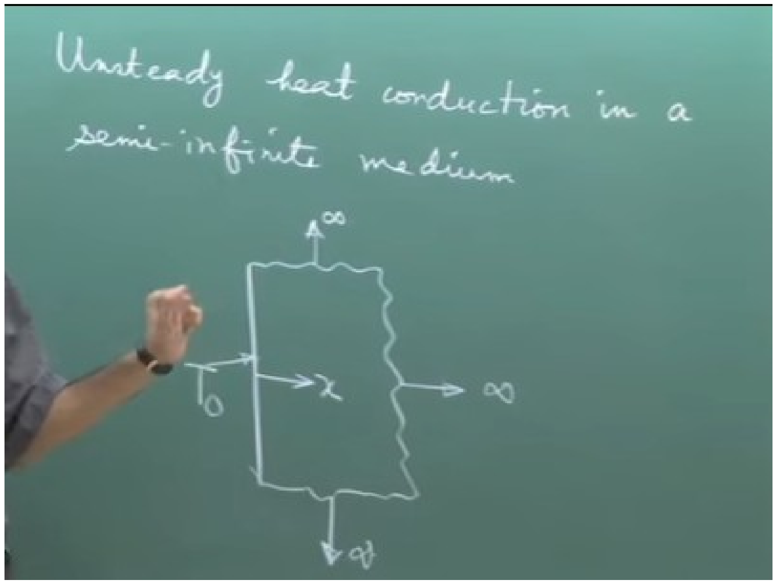

You have a solid medium like this which is also open to this direction which is x direction up to infinity but it is bounded in the other direction. That means, it is not open from minus infinity to infinity but open from zero to infinity. So, that is why semi-infinite medium. Now, question is like whenever we discuss some problem conceptually the first question that should come to your mind is that in engineering is there anything called as infinity.

Infinity is something which is a concept which we borrow from mathematics and basic science but in engineering can you say that let us make a wall of infinite thickness. So, that is not something we do in engineering. However, the term infinite has to be understood from a notional point of view and not from a literal sense. So, what do we understand by infinite or beyond what critical distance, it can be triggered as infinite although it is finite that is something which is a matter of importance.

So, we will see later on that. It is not literally infinite but what it means it that beyond the critical distance the or beyond the critical thickness of the wall the wall thickness can be assumed as infinity. So, we will try to assess that problem. So, just like the other problem these directions are also infinite so you have an one dimensional heat conduction along x. So, let us write the governing equation. Very similar to the problem of infinite plate we had this equation.

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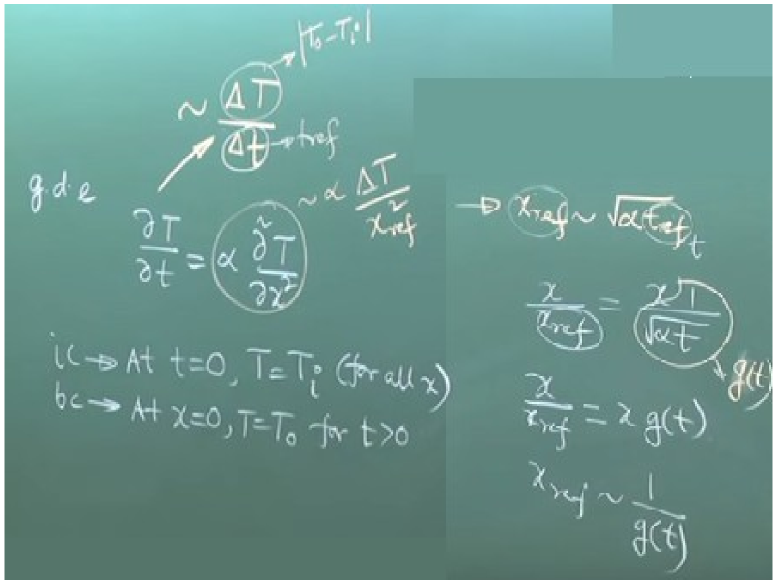

Which property has to be constant for this equation to be satisfied? K, Rho and CP, none of the above, tell? K, right. So, has to be constant. Now, out of other assumption are that the heat conduction in the y and x direction are not important that brings you to the one-dimensional problem. And you have to keep in mind that we are not considering any heat generation in this particular problem.

Now, we will solve this problem so let us write the initial and the boundary conditions as we discussed this kind of problem is called as initial boundary value problem in differential equation where you also have initial condition and you also have boundary condition. So initial condition At t equal to zero. This is the initial temperature and boundary condition. That at x equal to zero, this is x equal to zero, the temperature is T zero for T greater than zero.

So physical situation is there is block. This block was having an initial temperature. Suddenly, this wall is subjected to a temperature which is different from the initial temperature either hotter or colder and then there will be a temperature change within the block as a function of position and time that we have to find out. Remember that this is not the only possible boundary condition, you could have heat flux boundary condition or there are other possible boundary conditionals also.

But this we are solving as a representative problem to get the physical insight. But for other types of boundary conditions the mathematical solution may be different. But the physical insight will be very, very similar. Now can you tell, what is the most important difference between this problem and the infinite plate problem that we did in the previous class? What is the most important difference?

So, if you recall the problem of infinite plate that we did in the previous class this particular dimension we consider a fixed dimension of L the entire block was 2 A, half of the block was L. Now, for this we are not considering any fixed dimension. So, we are not considering any fixed dimension but we are claiming this to be infinite. Question is what possible type of dimension it can take for the claim infinite to be valid.

To understand, that we will do a little bit of order of magnitude analysis or scaling analysis. This term have an order of magnitude of delta T/ the time t. What is this? That in term of order of magnitude this represents a characteristic change in temperature delta T over a time, delta t. Now what is this typically? This is mode of T zero minus Ti. I mean, it depends on which one is higher and which one is lower?

But you have the characteristic change in temperature from the initial temperature to the maximum the boundary value which is provided. What is the order of the magnitude of this? Alpha, delta T/ x reference square, what is x reference? x reference is the distance along x over which the thermal effect is failed that means let us say you make this wall hot. Let us say, initially this block was at a uniform temperature now we make this wall hot.

If you make this wall hot there will be a distance up to which this heating at the wall will be failed. Beyond that distance if I'm standing I will not feel that the wall has been heated because the heating affect is not propagating beyond the critical distance. So, the question is how much the heating affect is propagated. Now, that can be obtained by noting that these two equations sorry these two terms in the equation they are balancing so they must be of the same order.

That means if you equate the order of this two then x reference (()) (10:58). This delta t we call as t reference just reference time of the characteristic time. So, what does it mean? It means that in this problem unlike the previous problem there is no natural physical length scale. The length scale is related to the time scale. This is the new (()) (11:39) that we get in this problem. In the previous problem while we were talking about the infinite plate the length scale was fixed by the geometrical length L.

Now in this particular problem it is not a fixed geometrical length that is the length scale of the problem. It is the length that depends on time that at what time you are considering the problem at that particular time there is a critical length up to which the thermal disturbance at the wall is felt. So that length is square root of alpha into the time that has elapsed. So, this is a variable length scale and the variable length scale depends on the time scale.

Now, none dimensionally it is possible to define a non- dimensional length. Let us call t ref as t, the local time t just symbolically. T reference is time when x is local x. So, this is a natural length scale of the problem expressed in a non-dimensional manner. And we will show later on the solution of this problem uniquely depends on this non-dimensional ratio. That means if you keep this ratio fixed then the non-dimensional solution is invariant. So, we will show that.

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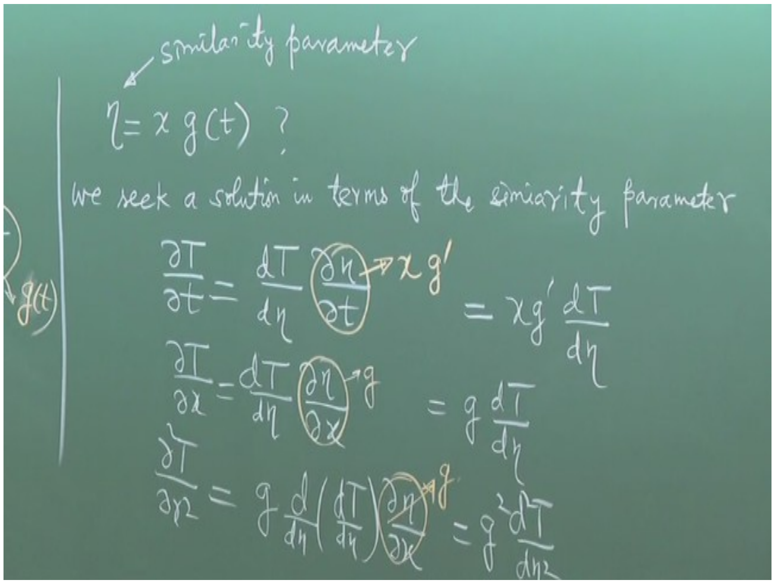

Now how do we show that for mathematical analysis we introduce the variable eta equal to x into a function of time. Question is where from our mind this is coming? This is coming from the fact that if this you call as or 1/this you call as g(t) then this is basically the solution depends on x into a function of time. So, see how physics guide mathematics, a mathematician will start from this form of solution but where from the form of the solution comes.

The form of the solution comes from the physical argument that the solution, if it is self-similar by itself then it should depend on x/x reference that is x by square root of alpha T and 1/square root of alpha t you can write as sum g(t). But the question is that this we already arrive from physical argument that g(t) is one by square root of alpha t or a multiple of one by square root of alpha t. But let us say that we do not keep that prejudice in mind.

So, can be show from mathematic also that gt will be of the same form as one by square root of alpha d. Let us try to do that so whether a similarity solution of this form this is called as a similarity parameter. So, we seek a solution in terms of the similarity parameter. See if you are by this time familiar with the method of separation of variables. It will appear to you that this is also a like a special case of method of separation of variables.

Separation of variable is like you can write it in terms of a variable which is a product of function of space into function of time. Here also it is like that but the function of space is the space variable itself and the entire solution is a function of this normalized variable. Question is can be express that or not that we have to figure out. You can always assume this to be a similarity parameter but for all problems similarity parameter does not exist.

So, we have to check mathematically whether the similarity parameter exists or not. So, let us see that how we can do that. So, if the similarity parameter exists then that entire solution is a single valued function of the similarity parameter that means it converts essentially the partial differential equation to ordinary differential equation. That means if this is a similarity parameter then the temperature will be a function of this parameter only.

So, will a function of beta only. So, delta T/delta T equal to dT/ d beta because T is the function of beta only and delta theta/ delta t. This is chain root. So, what is delta theta delta T this is x into g dash. g dash is bg dt. So xg dash dT/ d eta. We have to evaluate this term that on the right-hand side so delta T/ delta x. What is delta eta delta x, delta eta delta x is g. So, this is g dT/ d eta. The second derivative of this so this becomes g square.

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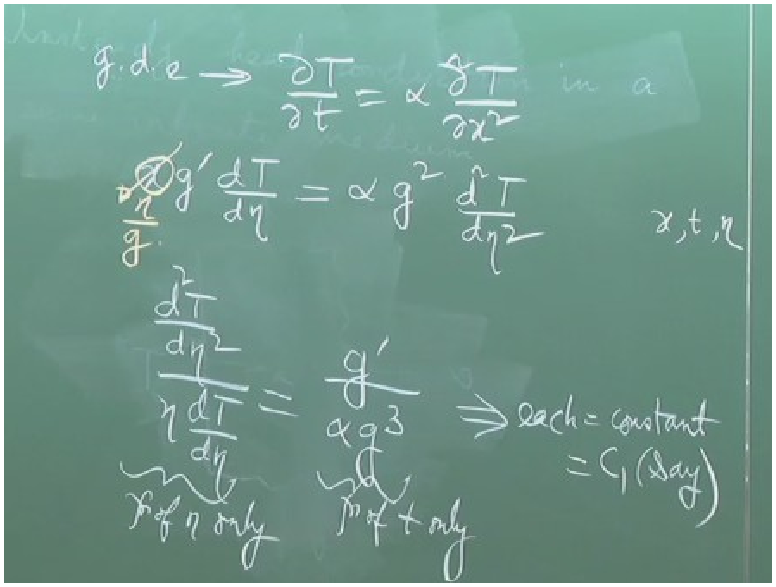

So, your governing differential equation to eliminate one of the variables. So, this equation involves three variables x, T and eta. Of course, T is dependent variable but I'm talking about the independent variables x, t, eta. So, to deduce the variables we will substitute x you can see x is nothing but eta/g. So, this you can write as eta/g. So, we have carefully written it this way because this is the function of eta only.

And this is the function of t only. So, function of eta only is same as a function of t only, only when it is a constant. So, if that we the case let us isolate this second equation and find out a form for g.

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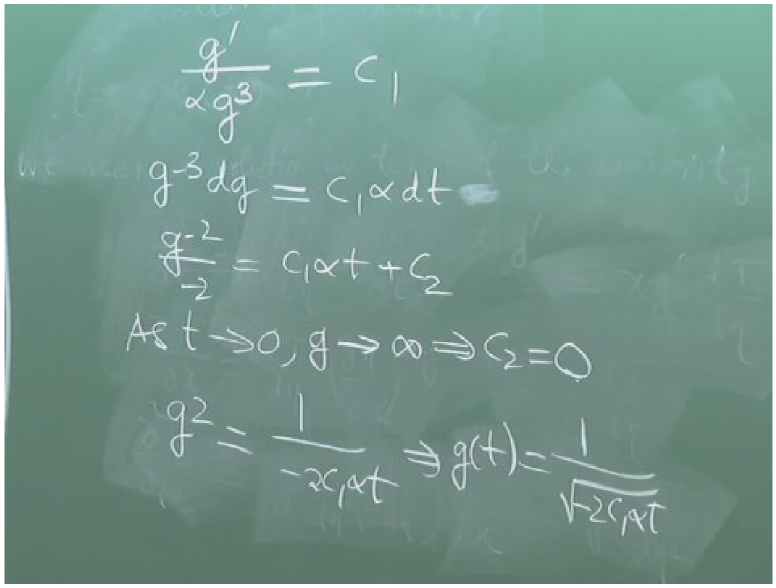

So, g dash by alpha g cube is equal to C1 so g to the power minus three dg is equal to C1 alpha dt and then you can integrate.