Conduction and Convection Heat Transfer
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Lecture - 26
Superposition Method for 2-D Steady State Conduction

We discussed about 2-dimensional steady state heat conduction and we will take it up from there. We discussed about one simple problem that you have a domain like this with these as the boundary conditions and this is the equation which we solved.

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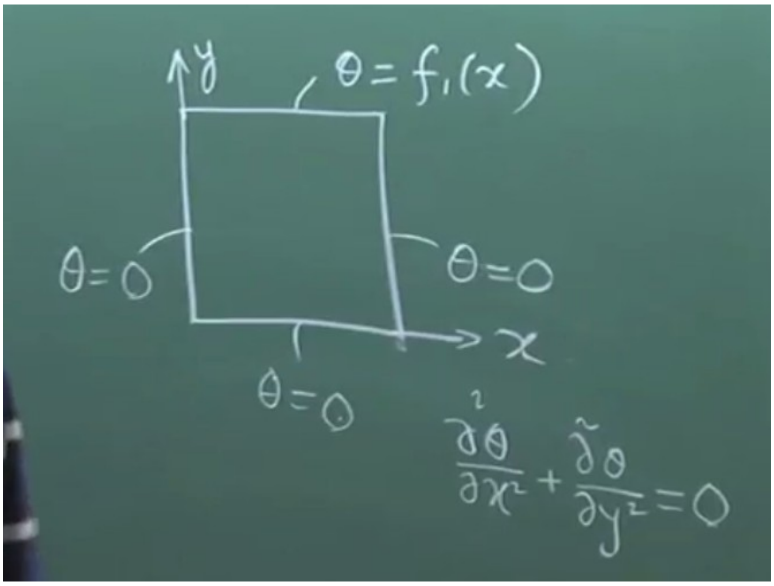

Now, we will try to develop some physical insight about this problem. So, to develop the physical insight let us take an example. Let us say f 1 x equal to 100 a constant, so in that case, if you consider this domain then how will the constant temperature lines are isotherms look like that is what we want to schematically show. So, these kind of physical insight is important because many times you can solve the problem and then find that graphically plot the solution.

But if you have a good physical intuition, good physical insight you can schematically draw the graphical representation without even solving the problem and that kind of skill is important for solving practical problems. So, now here you see that the temperature here is throughout 100, but here it is 0 in this boundary it is 0, so what is the predominant direction in which temperature gradient is created.

The predominant direction in which the temperature gradient is created is the Y direction because it is changing from 0 to 100 here. Along x also there is a gradient, but the gradient gives the periodic nature of the solution, ok. It is not a nonperiodic solution. It is a periodic solution along x and a nonperiodic solution along y. So that will mean that you will get constant temperature lines like this. What is the value of this? this is 100.

Let us say this is 50, this is 25 like that, ok. So, you will see that constant temperature lines will be like this. Now, there are typical situations when the boundaries are adiabatic and how will the constant temperature lines be at the boundary when the boundary is adiabatic. Let us consider a second example, this is example 1.

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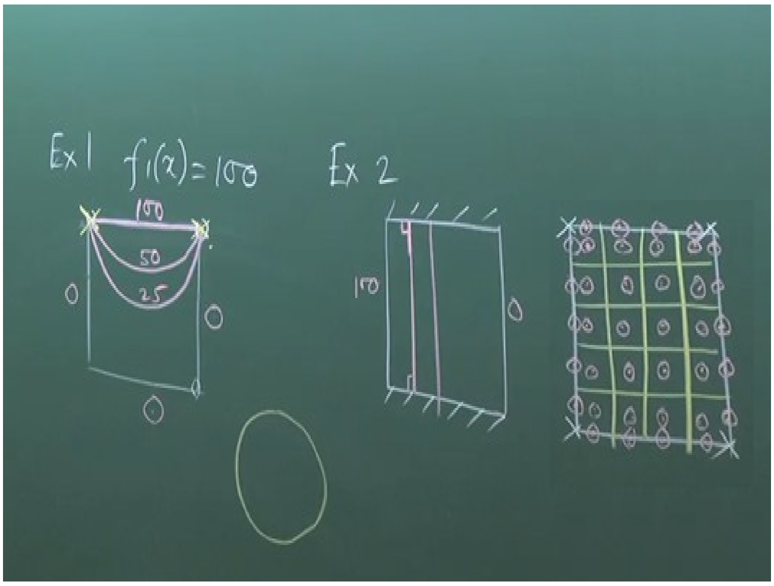

Example 2, let us say these 2 boundaries are adiabatic. When you have adiabatic boundaries, you will have constant temperature line perpendicular to the boundary and the reason is obvious. If it is adiabatic then what does it mean it means that the heat flex is 0 that means the normal temperature gradient is 0 that means along the normal direction temperature is not changing. Therefore, the normal direction is the isotherm or constant temperature line, ok.

So, this kind of a situation for example you will get isotherms like this. Let us say this is 100 and this is 0, so you will get isotherms like this. Now, all these examples give rise to some ambiguity and let us try to discuss about that ambiguity. Now let us look into this problem. Here the temperature is 100, here the temperature is 0, the shifted temperature that is theta is t- some t 0. Let us say this is 0, this is 0, this is 0 and this is 100.

And this is the question which we always like to avoid is that what is the temperature here or what is the temperature here right. This is the problem we always try to avoid and why we try to avoid this is because when the theories of differential equations are discussed over boundaries. The boundaries normally had not having any corner. They are smoothly connected boundaries.

Now, here you are having boundaries with corners and these are practical problems like in engineering you will often get boundaries with corners. So, the question is what is the temperature here. So, in mathematical analysis if you are solving the problem numerically then this kind of question is tackled in a different way.

So, if you have a domain, if you are solving this problem numerically, I told you that this belongs to the CFD for like solving these problems numerically but just to get some idea that what is done if you want to solve this problem numerically. What you basically do, you divide the domain into a number of subdomains, discrete subdomains and you may mark each subdomain with a grid point may be the centroid of each subdomain.

These subdomains are called sometimes elements, elements in finite element method, control volume, in finite volume method and so on. So, there are different names of the subdomains. So, if you have these points these are discrete points and then you also have discrete points on the boundaries because boundary condition needs to be incorporated, ok. There are some here also, but you can see that very purposefully.

These points are not considered to be parts of the discretization, right. So, what you are doing, you are trying to write algebraic equation instead of your differential equation. So, you convert your differential equation into a system of algebraic equations. Each equation for each of these circle points. So, as many number of circle points you have, so many number of algebraic equations you have.

Then you solve a system of algebraic equation by standard methods in linear algebra that is what you do but this corner points are not considered. So, why these corner points are not considered is because these are points of singularity. So, at this point the temperature is neither 100 nor 0, sometimes some people do it like this they say that ok let us make it 100 plus 0 by 2. There is everything in that except science.

So, that is not the proper way of handling it. So, these points are usually avoided because these are points of singularity and because these belong to 2 boundaries and these 2 boundaries have different conditions. Therefore, it is impossible to impose the condition at that common point. ok. So, typically this point is just excluded. So, excluding these points when you draw the isotherm it shows that this point is included.

But actually, this is slightly Epsilon distance away from the corner. Because in the limit that Epsilon tends to 0, so in drawing the figure we cannot show it, but conceptually it is a small Epsilon from the corner where the Epsilon tends to 0, but is not exactly equal to 0 because those are points of singularity.

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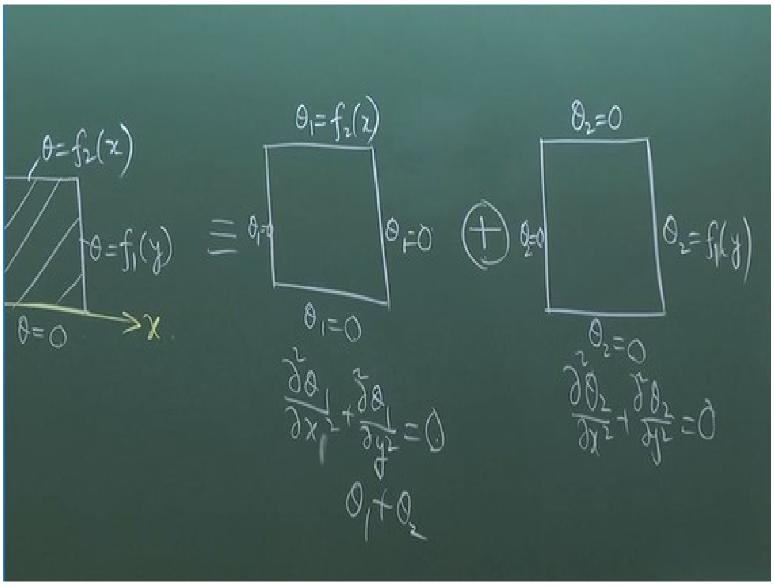

There is another example which I want to consider where we relax the boundary condition and consider a problem like this. Ok, you have a domain like this. This is your domain, you are interested to find out the solution for theta within the domain. Now, this clearly does not satisfy the requirement that 3 of the 4 boundary conditions are homogeneous. Here, only 2 boundary conditions are homogeneous and 2 are not.

So, these kind of problem can be solved by using the principle of superposition, so how this can be done. This is equivalent to 2 problems. Theta 1 is equal to 0, theta 1 is equal to 0, theta 1 is equal to 0, and theta 1 is equal to f 2 x plus theta 2 is equal to 0, theta 2 is equal to 0, theta 2 is equal to 0, and theta 2 is equal to f 1 y, so this will be governed by del square theta 1 del x square plus del square theta 1 plus del y 2 equal to 0.

This is del square theta 2 del x 2 plus del square theta 2 del y square is equal to 0 and the general solution is theta 1 plus theta 2. The reason is that because the governing differential equation is linear if theta is equal theta 1 is a solution and theta is equal to theta 2 is a solution theta equal to theta 1 plus theta 2 is also a solution to the governing differential equation which is shown here, ok.

So, this kind of superposition technique we can use for linear problems, we cannot use for nonlinear problems so in many problems were fluid flow is present then sudden nonlinearity may come or may not come, so if nonlinearity come then we cannot use this technique anymore, ok. So, this is about the 2-dimensional steady state problems. Now, we will give you some homework assignments and homework assignment will be uploaded.

And I would like to ensure that we send the homework assignments prior to you over email and like for the mock course it will be uploaded in some sites and then there will be a separate tutorial when we discuss about the solutions of the specific problems. So, the way in which we go about this course is that we work out some problems in the class, we give you some homework problems, out of which some of them have might have already been worked out in the class or somewhat straight forward extensions.

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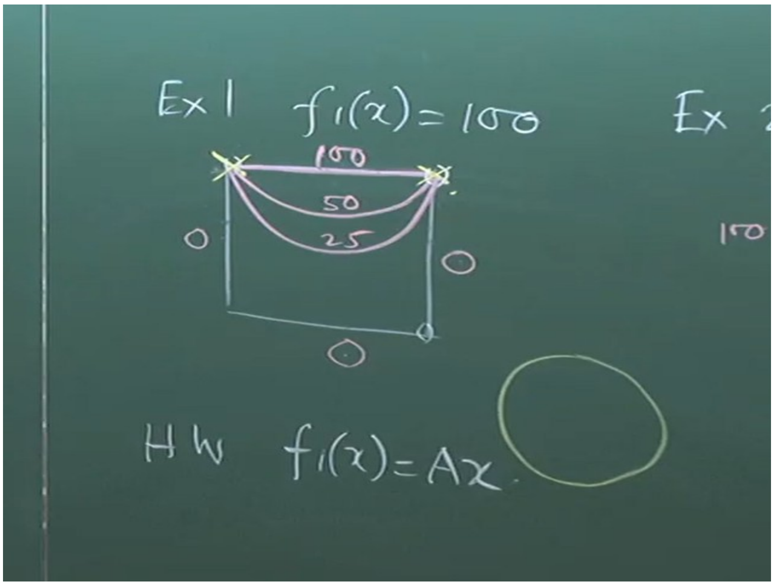

For example, I can give you a homework problem where f 1 x is equal to a x. This is the problem from the exercise of the textbook of (()) (16:03). So, f 1 x is equal to a x, now it is only a simple integration that you have to do because the entire frame work I have developed already in the class, so in heat transfer remember sometimes students say where is the problem because it is very difficult to understand what is theory and what is the problem.

In heat transfer, everything is theory and everything is problem. So, there is no distinction between it is not like a junior school level thing, so that you have a formula you monitor and then you put a value in the formula it becomes a problem, so that kind of education in heat transfer we are not going to provide you.

So, the problem means it is basically from the scratch, from the first principles you have to derive and if there are numerical values you just have to plug in the numerical values but we do not expect that you remember any formula to solve any problem that is the kind of philosophy that we will adopt in the course of heat transfer, so no formula based study please, ok. Now, we will move on to our next issue in the conduction heat transfer, which is unsteady.

Of course, we have done steady state problems, so the next obvious extension is unsteady, but there is something in between which is neither steady nor unsteady and that is called as quasi steady and I will discuss some such problem before coming into the unsteady problem.

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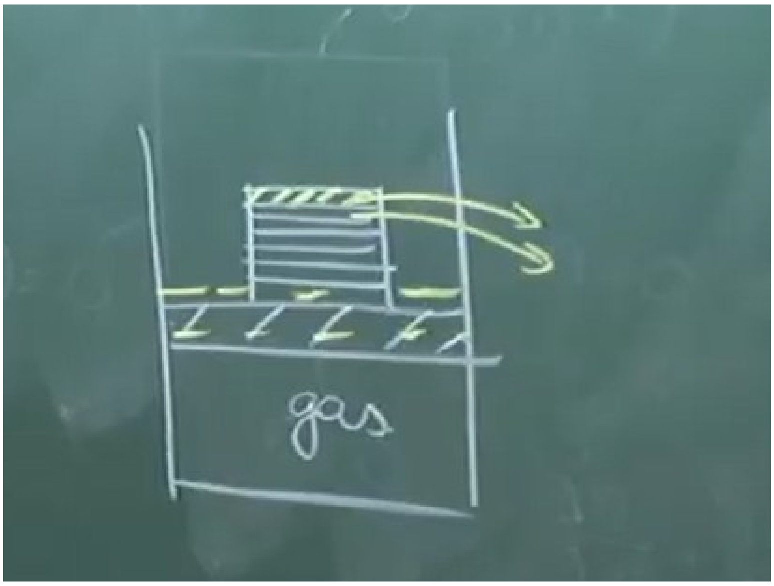

So, quasi steady has some resemblance with quasi-static or quasi-equilibrium process that you must have studied in thermodynamics, so again the only distinction is that there is a difference between steady and equilibrium, so that much of difference is there but notionally if you think of a problem in thermodynamics, a classical situation when there is a piston moving in a cylinder, very classical problems in thermodynamics involve a piston moving in a cylinder may be there is some gas in between the piston and the cylinder.

Lets us say that you have some waves on the top of the piston, the waves are such that these are very thin slices so what will happen you remove this top slice this piston will go up by a little bit, right. Then, you remove the next slice the piston will go further up by a little bit because for a given mass the pressure is decreasing, so the volume is increasing right.

So, in this way it will undergo a very slow expansion such that for the entire expansion process all the in between states are almost in thermodynamic equilibrium and the deviation from thermodynamic equilibrium is very little. So, the keyword is that this is the very slow process because it is a very slow process the deviation from equilibrium is almost nil, so that kind of process is called as quasi-static or quasi-equilibrium process in thermodynamics.

So, similar analogy in heat transfer not exactly the same again I am giving you a caution there is a difference between steady state and equilibrium which we have discussed, so similar analogy in heat transfer is a quasi steady. Quasi steady means in the long run it is unsteady because things are changing with time, but the change is so slow that the entire change can be thought of as a collection of a change taking place through a large number of intermediate steady states that kind of process is called as a quasi steady process.

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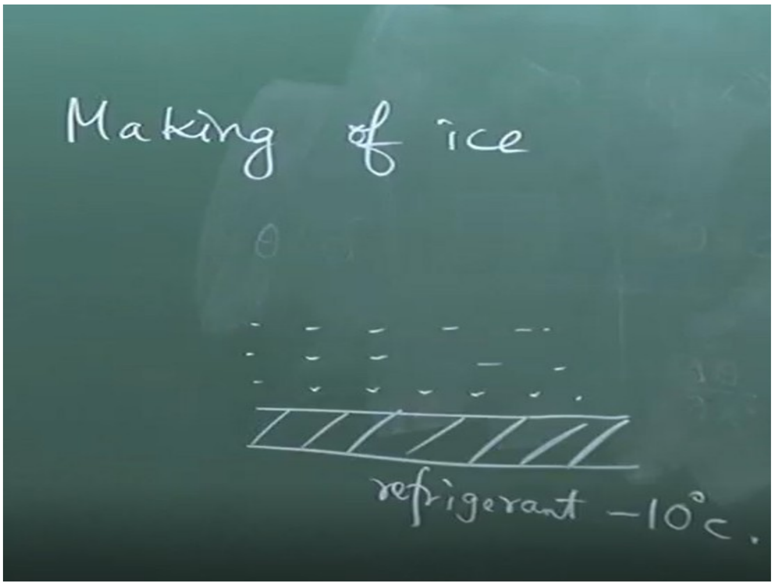

So, now I will discuss about an engineering problem as an example you may think of it as a problem or as an example or whatever, let us say making of ice. Ice making is a big industry and there are many ways in which ice can be made. Now, there is a situation something like this you have a metal plate and there is a refrigerant which is flowing below the metal plate with a particular velocity and there is a heat loss, so over this there is water.

Let us say this refrigerant is at minus 10 degree centigrade, ok. So, this water because it is at a higher temperature as compared to this, it will lose heat through this metal plate by conduction and once the heat is lost it will come to a state when the temperature comes at the freezing point of water and then some ice will be formed. The water converted into ice its layer will thicken and beyond the critical thickness that is crept off and new layer of ice starts forming.

So, this is a simple old-style technology of making ice, ok. So, now let us make a heat transfer analysis for this particular problem. So, we can make a quasi steady type of analysis but this is a very interesting problem where we are involving a change of phase. So far in the theoretical description of heat transfer, we have purposefully avoided problems with change of phase. Now, let us bring into an example where we have a change of phase.

So, let us say that this is an interface, ok. On this side there is solid, on this side there is liquid. I am not discussing about this problem first, but I am discussing about a different problem just to know that what kind of boundary condition you should use at the interface between the ice and the water.

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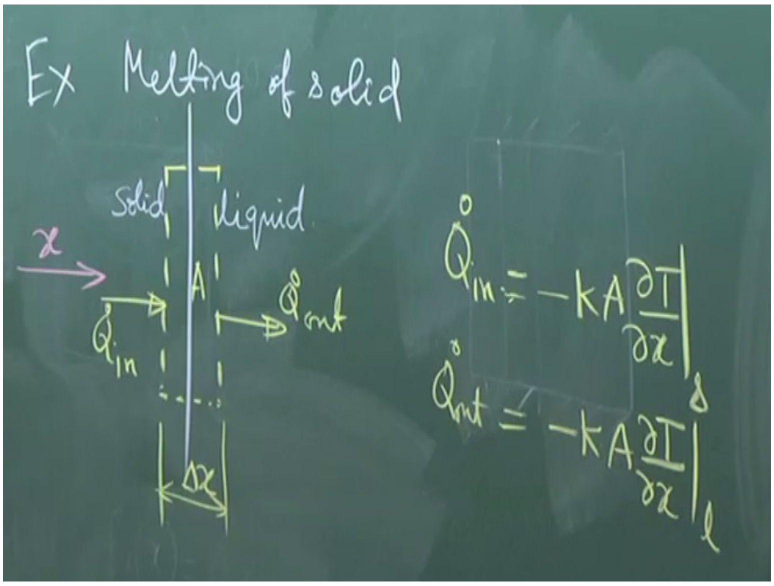

So, we are taking an example which is little bit different this is melting of a solid. So, let us say there is a solid, there is a heat transfer here, let us say it is a 1-dimensional situation. There is a heat transfer along x, so some heat is utilized for melting the solid to liquid. Let us say that we take a small control volume like this surrounding the interface. This control volume is of thickness delta x in the limit as delta x tends to 0 we can recover a sharp interface.

In reality, in the molecular vault no interface is sharp because there are molecules in one phase and molecules in the other phase and there is a gradual transition when you go from the molecular arrangement of one phase to the molecular arrangement of another phase. So, it is not sharp but macroscopically if you see it will appear to be a sharp interface and that sharp interface is recovered in the limit as delta x tends to 0.

But we take a control volume like this and let us say that there is a rate of heat transfer Q dot in and there is a rate of heat transfer Q dot out. Let us say A is the area of the interface perpendicular to the direction of heat transfer, so what is Q dot in this is minus KA by Fourier's law. What is Q dot out similar term. Now, which one is more Q dot in or Q dot out, some heat has been transferred to the control volume.

And something is leaving now in between what has happened in between the solid has converted into liquid, so what has been the case when the solid has converted into liquid some heat has been taken by the solid to get converted into liquid in the form of latent heat, so whatever is going out must be less than whatever has come in because whatever is going out and whatever has gone in a part of that is utilized for melting.

So, whatever is going out is less than whatever has gone in.

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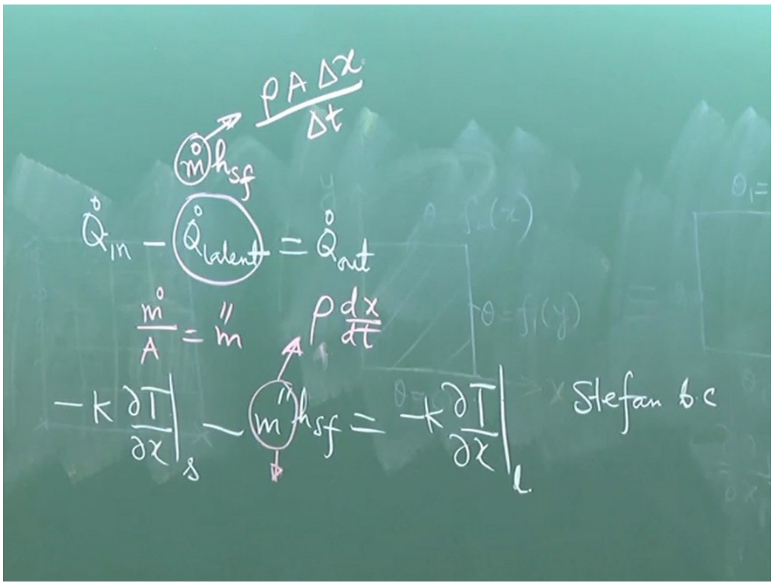

So, we can say Q dot in minus Q dot latent is equal to Q dot out, right. So, what is Q dot latent, so M dot into the latent heat which is let us say h s f, ok. We will see what is M dot but I will just symbol wise this is the rate of mass being converted from solid to liquid and that multiplied by the latent heat is the total latent energy transfer, ok. Now, if you divide all the terms by the cross-sectional area then M dot by A let us say that is mass flex M double dash then we can write minus K.

Of course, what is M not actually the row into A into delta x this is the mass divided by the time delta t over which the space change process is being studied. So, M double dash is that divided by A, so this becomes row into limit as delta t tends to 0 delta x by delta t right because why we make the limit as delta t tends to 0 because in the limit of delta t tends to 0 delta x will also tend to 0.

Then only this interface will become a sharp interface macroscopically, so this will become ok. Now, there is something which is more interesting at this undergraduate level I do not want to bring this issue, but just an open question which you think about this is row of which? row of liquid or row of solid. Yes, row of liquid or row of solid. So, sometimes you know when you given an answer in the answer book you write something here.

Then if I say that it is liquid you say I have written liquid, you cannot see it and because as professors we are old with spectacles and all then it is expected that we cannot see it, so you can keep it fuzzy but heat transfer is not a topic of fuzzy logic it is different. Fuzzy logic is a different subject and heat transfer is a different subject, so there is no fuzzy answer to it. I will complicate the question by even one more standard what is that let us say that the density of the solid and liquid is different and that is practical.

Usually if a substance melts either it will expand or it will contract, so the densities are different then how is this differential density taken into account in this, ok. These are 2 questions which I keep open for this particular case we will consider the density of the solid and the liquid phase are equal so that that ambiguity is not there, so you can consider this is as either row S or row L.

But I will ask you the answer please think about it, it is a question for thinking. Now, this type of boundary condition in heat transfer is an interfacial boundary condition known as Stefan boundary condition, ok. So, you can use this Stefan boundary condition for heat transfer for a phase change, any phase change. If it is evaporation, condensation just these things will change.

For evaporation, this will become h f g. From saturated liquid to saturated vapor whatever is the latent heat. If there is no phase change then these term will be 0, right. So, at the interface what is the boundary condition that you have minus K del t del x is continuous. Why because interface cannot store thermal energy, so whatever thermal energy has come to the interface the same thermal energy should be transferred to the other side of the interface.

So minus K into temperature gradient normal to the interface that is continuous. What is the analogy in fluid mechanics, if you have flat interface and let us say that you have a velocity profile along Y then mu, du, dy is continuous across the interface. So, if you have liquid and vapor, so mu liquid into du, dy at the liquid is same as mu vapor into du, dy at the vapor. Of course, this will be little bit disturbed if you have a curved interface and so on.

But I am not bringing that complication at this level. I am just trying to give you an analogy between heat transfer and fluid mechanics.

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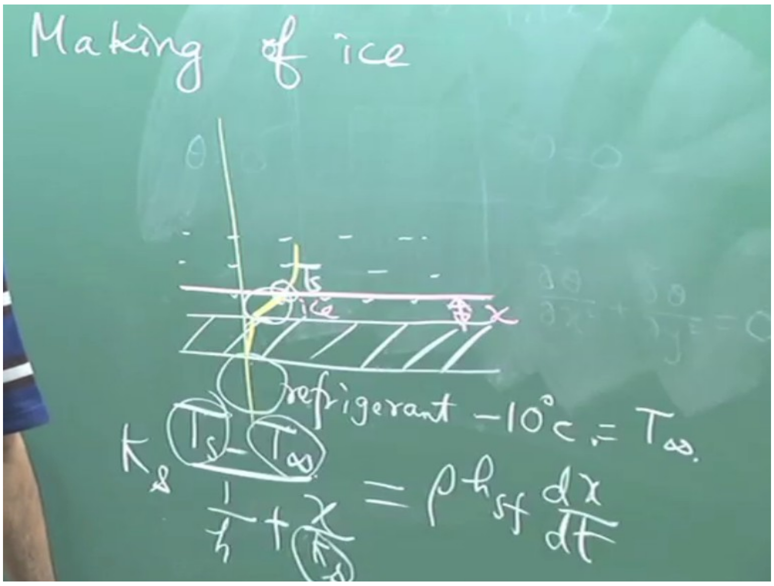

Now, how do you apply that formula to this particular problem. This problem is just inverse of this problem where instead of melting it is freezy, ok. Now let us try to assess the problem by assuming that this is the layer of ice that has been formed and let us say the thickness is x. So, now can you draw the temperature profile within the system, so first of all this is a metallic plate, why do we use the metallic plate in this case in engineering?

See, metallic plate will have a very high thermal conductivity, so the temperature drop within the plate is very small or temperature rise rather in this case not drop. Temperature rise within that is small, so that means virtually the water is exposed to a temperature which is close to minus 10 without any substantial increase within the metallic plate, ok. So, if you want to draw a temperature profile.

So virtually there will be some little change because it will have some conductivity then within the ice, it is pure 1-dimensional conduction, so there will be a linear temperature profile and then within water the temperature gradient will not be as sharp as that in the ice. So, you can safely neglect this turn. The temperature gradient in the liquid phase is much much less than the temperature gradient which is there because the temperature gradient is virtually is primarily imposed across this, ok.

So, you are left with these 2 terms, this term and this term. So, let us say that the temperature here is T s and the temperature this minus 10 degree centigrade is T infinity. So, how do you write this term you write K of solid into the temperature gradient, so basically you write T s minus T infinity. Now when you write T s minus T infinity you have to take the resistance of this phase plus the resistance due to convection here.

The resistance due to conduction here is very small because the conductivity of the plate is very high. So, you have basically 3 resistances in series, one resistance is because of the flow of refrigerant, this refrigerant is flowing so what kind of resistance it is creating it is a convective resistance. So, let us say H is the convective heat transfer coefficient here and this is the conductive resistance due to of course the area is observed here.

Because it is already divided by area that is why the area is not put there. This term balances the other term that is row into h s f into d x, d t, right. So now you can integrate this with respect to x to find out x as the function of time, ok. So that is a simple state forward integration, I am not putting that here. So, this will tell you that how do you design the system.

Because if you want to design the system you have to design that what is the thickness of the ice that is formed at a given time because that is your productivity so you must keep a proper design and you will see that what are the parameters which are defining these. What is the temperature here, what is T infinity and what is the conductivity of the solid and what is the heat transfer coefficient?

Heat transfer coefficient will be higher if you flow the refrigerant at a higher speed, so that is where convection will come into the picture, ok.