Conduction and Convection Heat Transfer

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Lecture - 25
Separation of Variables Method for Two-D Steady State Conduction

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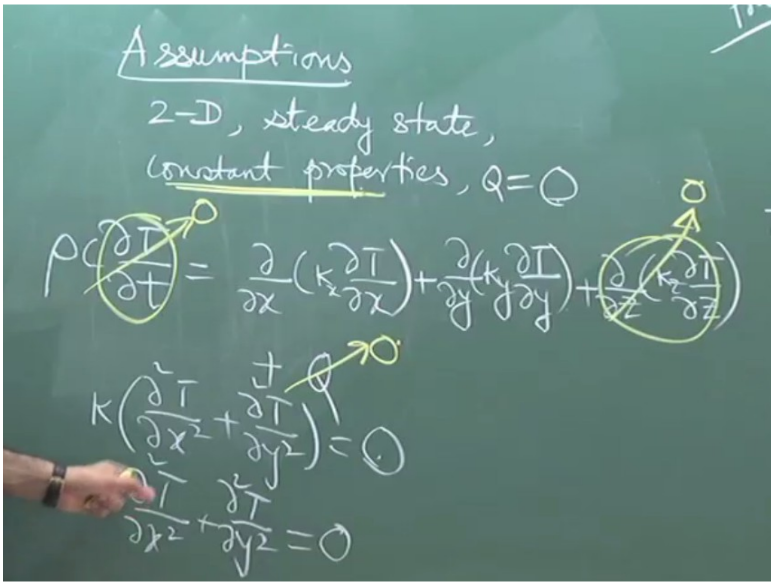

This prototype is not just important in heat transfer, but in any branch of applied physics and engineering and this is known as Laplace's equation. Of course, this is Laplace's equation in 2 dimensions. If you have another term like this for z that is Laplace's equation along 3 dimensions. Now we will try to solve this equation. Now there are many methods of solution and each method is very interesting by itself.

These bring the most elementary level course on conduction heat transfer. We will not start with very elaborate method of solution. We will start with one of the simple, but very insightful method of solution which is called method of separation of variable, but method of separation of variable will not always work and it works we will see I will derive the equations in detail and my teaching philosophy is very simple.

Whatever I cannot derive at your level, I will not teach. Whatever I can derive at your level and make you understand that only I will teach because that is what what you will learn. Not that you learn thousands of formula and reproduce without understanding where from it comes that is not the education that I want to impose on you. So, I will consider a simple case without any prejudice in mind we will start and we will solve the entire problem.

But I would like to let you know that method of separation of variable does not work always. So, when method of separation of variable does not work always. When does it work and when it does not work? We will look into it in detail but one of the important requirements like here how many boundaries are there? There are 4 boundaries right in this domain. So, in this kind of a problem, it is important that out of these 4 boundary conditions at least we can map 3 homogeneous boundary conditions.

So, what is a homogeneous boundary condition? Homogeneous means right hand side equal to 0. I mean of course this is not a mathematical definition of what is homogeneous, but this is a practical way of looking into it. Something right hand side equal to 0 that we call as homogenous. So, here T 0 may not be 0, right?

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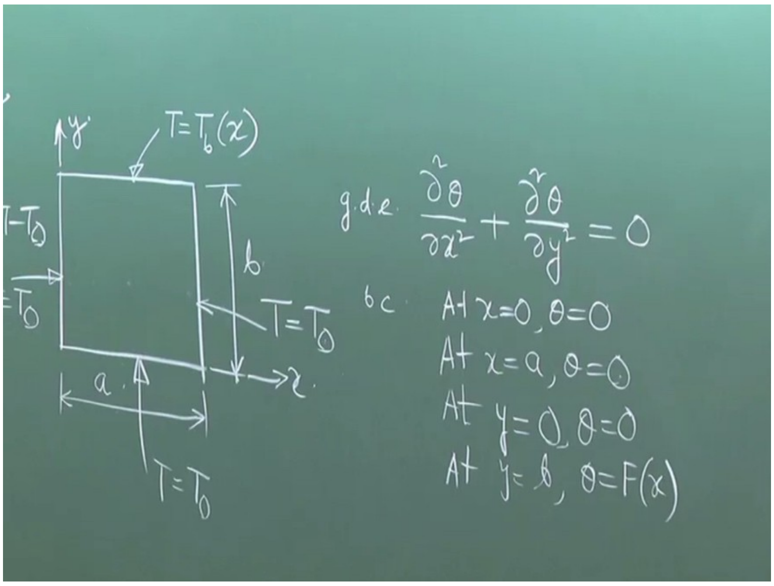

So, to convert this to 0 what we can do? We can just redefine; we can make theta is equal to T minus T 0 right. We can define in this way. See that is why deliberately these kinds of boundary conditions were given. I will show you that is 1 of this is relaxed how will you do? Ok so let us say this is T 0, this is not equal to T 0. How to solve that problem? I will give you some idea later on.

But, you can use a technique called as super position to solve that problem using the method of separation of variable itself. But, we will start with the simple one where we define theta is equal to T Minus T 0. The reason for defining this is to make all these 0 as boundary condition. So, the governing equation does not change. So, in separation of variable, what we assume? So, let us define the problem properly.

So, this is the governing differential equation. Boundary condition - so let us setup the x and y axis:

1. At x is equal to 0, theta is equal to 0.
2. At x equal to a, theta is equal to 0.
3. At y is equal to 0, theta is equal to 0.
4. And at y is equal to b, theta is some function of x.

So governing equation and boundary condition - it defines the physical problem.

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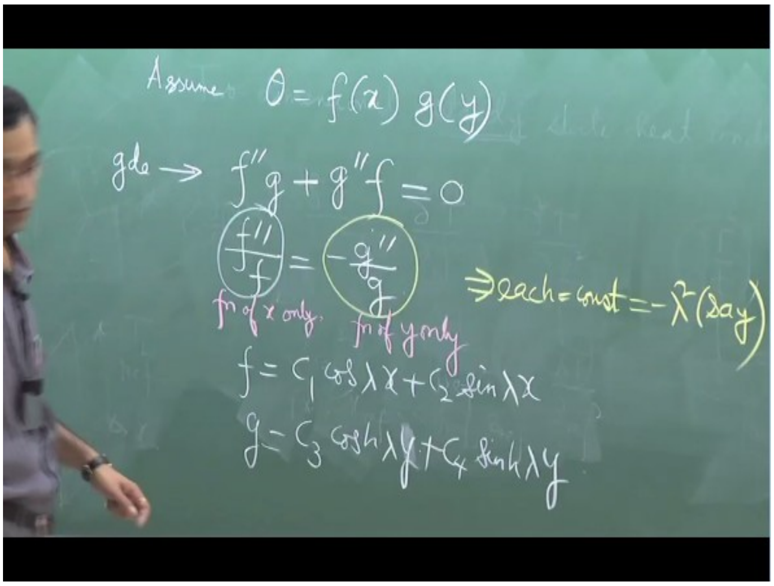

So, in separation of variables, what we do? We assume again I am repeating that in all cases you cannot assume this. It depends on the boundary conditions and we will see for what type of boundary conditions you do this. If you cannot assume this. What are the other techniques? There are other analytical techniques somewhat more cumbersome, but it is more straight forward to solve this problem numerically.

If you want to know how to solve this problem numerically for that you have to study "Computational Heat Transfer and Fluid Mechanics" or CFD. Now, I mean in all the books of "Heat Transfer" there is some elementary computational or numerical parts. If you are interested you can look into that but, I believe that is too elementary. If you are serious about learning how to solve this equation numerically get hold of any CFD book.

Or you can consider the online video lectures of CFD. There is one NPTEL video lecture of CFD by me. You can consider that if you are interested. That how to solve these equations numerically, but I am not getting into that here. In most part of this course, we will try to go for analytical technique. These are classical mathematical techniques which will give you a good grasp on the mathematics of the problem, which is necessary for many physical insights that you want to develop.

So, theta is equal to f x g y. So, from that equation what you (()) (35:45) So the second derivative with respect to x will mean f double dash g right. Dash is derivative with respect to x plus g double dash f is equal to 0. This is straight away from the governing differential equation. So, you can write f double dash by f is equal to minus g double dash by g right. See if somebody ask you that why do you go for method of separation of variables?

You should not give this answer because this was taught to me in the class that is why I am interested about method of separation of variables. So, method of separation of variable has one big issue that is why it is so important. It allows you to convert a PDE to a ODE. It allows you to convert, if it is applicable it converts a Partial Differential Equation to an Ordinary Differential Equation.

And there are much more elaborate and commonly available techniques for solving ODEs than PDEs. So, converting PDE to ODE there are several other techniques. We will see later on that there is a method called a similarity transformation that also does it. We will look into it when we solve unsteady problems, but the whole idea is that there are whole classes of problems or whole classes of mathematical techniques which allow you to convert the Partial Differential Equation to Ordinary Differential Equation.

So, this is one attempt to do that. How we do that? This is a function of what? x right. So, this is function of x only. This is a function of what? y only. A function of x only is equal to a function of y only for a general function. It is possible only when it is a constant right. Otherwise, it is not possible. So, this implies each is equal to constant is equal to minus lambda square say.

So, question is and a very big question is that why do we take this as minus lambda square and not plus lambda square, right. Always there are 2 ways of learning, 1 way of learning is I mean somehow you have got the idea. Another way of learning is learning through hard way and that is not very bad actually. Sometimes we undermine it, but it is not very bad. So instead of minus lambda square you may substitute plus lambda square.

And see that what kind of mess in which you come up while solving the problem, but that is the hard way of learning. But an insightful way of learning is also there. So, when you substitute it as minus lambda square. See what kind of solution you expect along x? Physical intuition. For this problem what kind of solution you expect along x? You expect a periodic solution along x because it is repeating again right.

So periodic means you have sine, cosine type of variation. So, if you put minus lambda square here, f double dash plus lambda square f equal to 0 will give sine and cos as the solution for it right. So, f will be C 1 cos lambda X plus C 2 sin lambda X right. This is straight forward homogeneous second order Ordinary Differential Equation in high school level you have done this. Then what about g? So, g will be exponential right.

E to the power x type of form, e to the power lambda X and e to the power minus lambda X. Sometimes for easy mathematical manipulation we instead of using e to the power x we use sin hyperbolic and cos hyperbolic x. Those are also linear combinations of e to the power x and e to the power minus x. Like you have solved fin problem. You have seen that in problems in fin you could write the solution in terms of e to the power mx and e to the power minus mx.

But usual it is written in terms of sin hyperbolic and cos hyperbolic just for mathematical manipulation. You can also write in terms of e to the power, there is no problem with that.
So, g is equal to C 3 cos h lambda Y plus C 4 sin h lambda Y. Now we will try to apply boundary conditions and solve for this constants to the extent possible. Let us give some numbers to the boundary condition. This is 1, this is 2, this is 3, this is 4.

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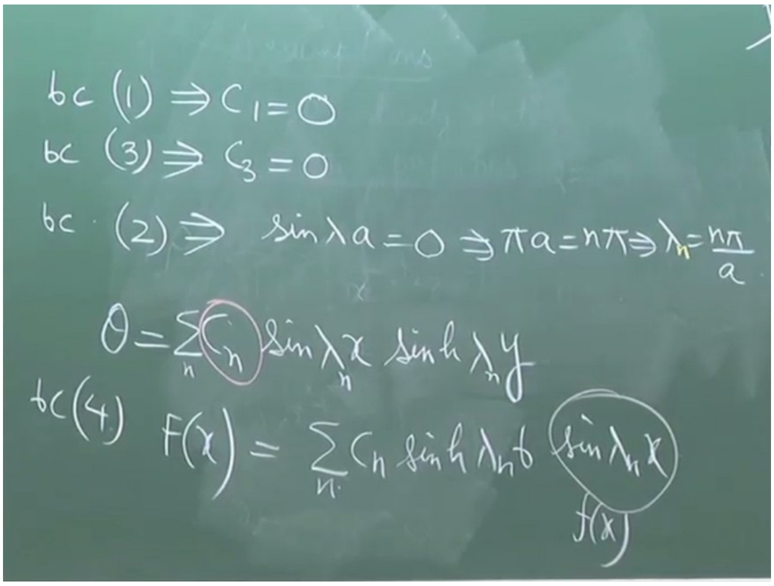

Boundary condition (1): At x equal to 0 theta equal to 0. Theta equal to 0 means f equal to 0. So, at x equal to 0 f equal to 0. So which term will be zero? C 1 will be 0 right. Now we apply the boundary condition number (3): At y equal to 0 theta equal to 0, so that means C 3 equal to 0, ok. So, the solution has got read of the Constance C 1, C 3. You have 2 more. Now you apply the second boundary condition (2):

At x equal to A, f equal to 0 means C 2 sin lambda A equal to 0 right. So, there are 2 possibilities, one is C 2 equal to 0 and other is sin lambda A equal to zero. What we do not take C 2 equal to 0 why? If we take C 2 equal to 0 the total f becomes 0 and theta becomes 0 that is a trivial solution. A trivial solution is theta equal to 0 everywhere. So, we are interested for a non-trivial solution and so for non-trivial solution sin lambda x equal to 0.

This is called as IN value problem in differential equation. Just like you have IN value problem in linear algebra, so for non-trivial solutions you have certain lambdas which give non-trivial solution to a system of algebraic equations. Here similarly you have a non-trivial solution not algebraic equation, but of a differential equation. So, these are called as IN value problems in differential equations.

So, you have sin lambda A equal to 0. So, lambda A is equal to n pi. So, lambda is equal to n pi by A. So how many possible values of lambda are there? Infinite number of possible values depending on infinite number of possible values of n. So, to mark that we sometimes give a subscript n to lambda to indicate that there are numerous infinite number of values of lambda.

So, theta when we write, so it is basically C 2 into C 4 let us call it C n sin lambda X sin h lambda Y. What is C n? C n is C 2 into C 4. Now for each value of lambda you get a solution like this. How many such cases are there? Infinite number of such values of lambda are there. So, you will get infinite number of such terms and the general solution is the sum of all those terms why? That is because of the linearity of the governing differential equation.

If theta is equal to theta 1 is the solution and theta is equal to theta 2 is the solution then theta is equal to C 1 theta 1 plus C 2 theta 2 is also a solution for this. So, in this way for lambda equal to lambda 1 you have a solution lambda 2, lambda 3 in this way all those terms sum together will also be a solution. That is because of the linearity of the governing differential equation ok. So, this is summation over n.

So now our task remains that how to find out this C n because once we find out this C n then that completes the solution of the problem. Only one constant means one set of constant because C n will be a function of n of course. So now we have only one boundary condition that we have yet not used. So, the boundary condition number (4): Theta is equal to f x at Y equal to b so summation of C n sin h lambda n B into sin lambda n X.

Remember this is what is f x. Now how to get this C n. So, we will see how to get this C n. Let us consider this equation, which I have marked by arrow.

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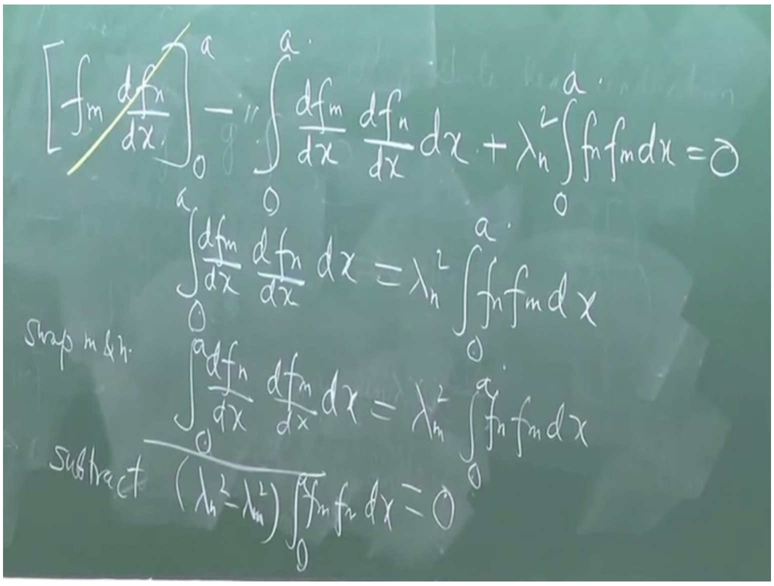

So, this equation just I have expanded it in full form and used a subscript n, this equation. Now what we do is we multiply this equation by f m, where m is not equal to n or a special case is there when m equal to n that is one value. In general m may not be equal to n, ok. So, we now integrate it with respect to x from x equal to 0 to x equal to a. So, when we integrate this we will use integration by parts and quite obvious because in integration by parts.

We have first function into integral of the second function. Second function we normally keep higher order derivative. Because when we integrate, the order of the derivative comes down. So, this is the first function. This is the second function. So, first function into integral of the second minus integral of derivative of first into integral of the second. Now I apply the boundary condition. So, this is the boundary term right.

What is the boundary condition? At x equal to 0 theta is 0 right. So, f equal to 0 and at x equal to L also theta is 0. So, you can see this boundary term goes out. Had it been that at one of the boundary instead of theta is 0 it is d theta d x 0 then also this term would have been 0 ok. So, this term has become 0 and you are left with integral d f m d x into d f n d x. Now you swap m and n.

So, if you do that you will get just swap m and n, ok and subtract that which you subtract you will get lambda n square minus lambda n square, ok.

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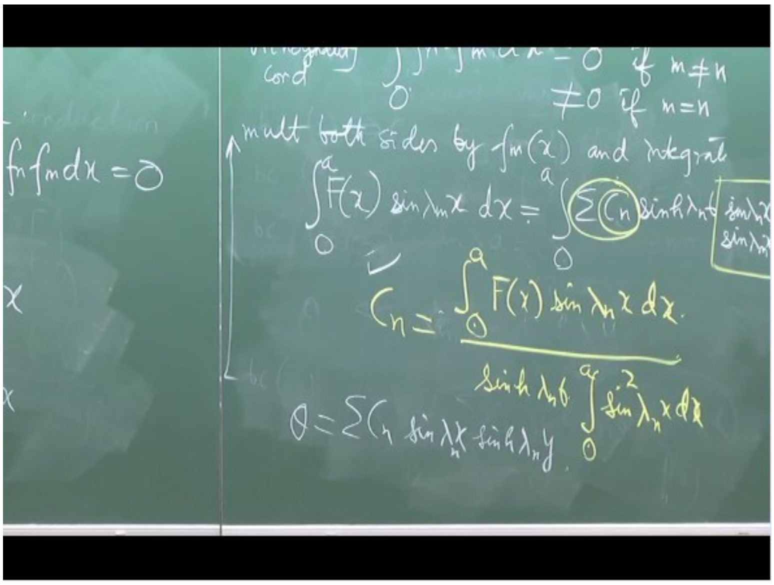

So, from here what can we conclude? We can conclude that integral f n f m dx from f equal to 0 to f equal to a. This is equal to 0 if m not equal to n right. In general, to satisfy this if m not equal to n this term is not 0. So, to make this 0, this is in general 0. But if m equal to n that is not necessary. So, this is not 0 if m is equal to n. This is called as orthogonality condition. So why such a name orthogonality condition out of Tom Dick Harry why orthogonality condition?

So, you are familiar with orthogonality of vectors right. When you have 2 vectors, when do we say that 2 vectors are orthogonal to each other? When their dot product is 0 right. They are perpendicular to each other. So, here in a functional space instead of dot product we talk about inner product and this is here instead of orthogonality of vectors it is orthogonality of 2 functions.

So, when you say dot product of a vector that is a special case of inner product. Basically, you are multiplying 1 component of 1 vector with the corresponding component of the other vector and adding it out. So here also you are multiplying the function with its corresponding n index with m index. So, when this is 0 then we say that it is an orthogonal function. So, we will use that orthogonality here. So how can we use the orthogonality here?

This is f and x right. We will multiply both sides by fm and x and integrate. So, what we will do multiply both sides by f m x and integrate ok. So now the charm of using the orthogonality condition is that this summation will go away and you will be able to isolate C n because the integral is essentially integral on this function and the integral will be 0 for all m not equal n only for 1 term when m equal n the integral will be non-zero.

So instead of the summation the C n itself can be isolated. So, you will have C n is equal to integral of 0 to a f x sin lambda n x d x divided by sin h lambda n b integral of sin square lambda n x d x, that completes the solution of the problem. So, this integral you can very easily evaluate you substitute lambda n a equal to n pi that was the value of lambda n right. So, lambda n is n pi by a that you can substitute here and here sin square theta you can write half into one minus cos 2 theta and integrate.

Very simple integrations are there ok. So, once you get the C n then basically you can write the solution in terms of an infinite series theta is equal to summation of C n sin lambda n x sin h lambda n y. I tried to do it from as fundamental as possible without referring to the Fourier series. But you can if you use your familiarity with the Fourier series you will see eventually the terms in the series that come, they quite clearly resemble with the terms in the Fourier series ok.

So, to summarize what we have studied today? We have discussed about what is the physical way of assessing whether a problem is 1 dimensional or 2 dimensional or 3 dimensional steady or unsteady and then we have invoked simple problem of 2-dimensional steady state heat conduction and we have solved that problem. So, we stop here today and we will continue in the next class. Thank you.