Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 54
Navier-Stokes equation- Part – IV

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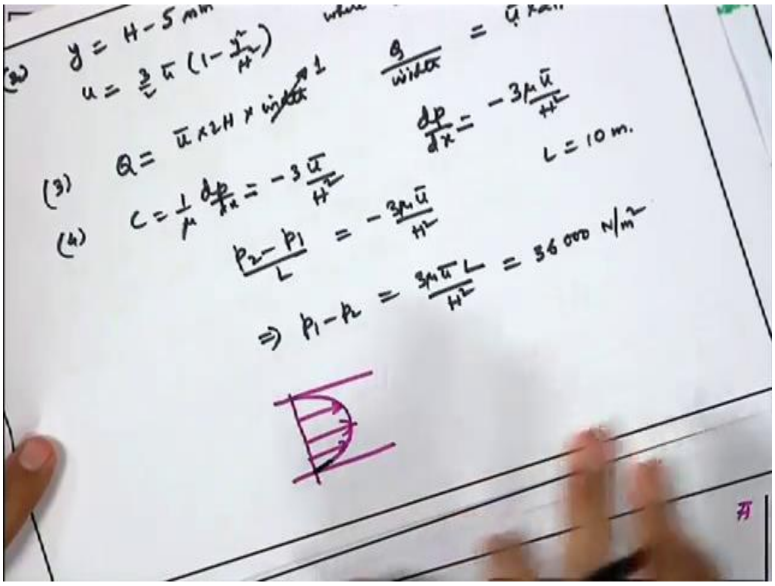

So, we will consider a second example of exact solution of the Navier-Stokes equation through something which is very important in fluid mechanics this is called a Couette flow.

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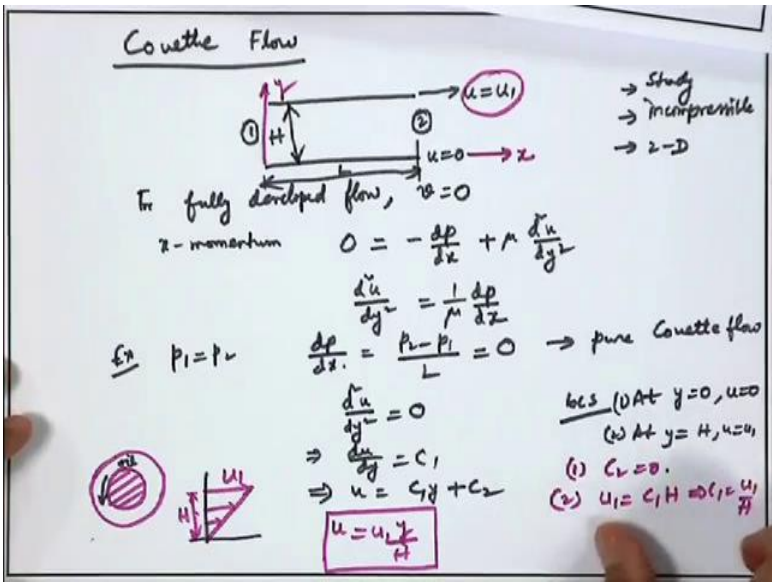

Consider two parallel plates with relative motion; that means, let us say that the bottom plate is stationary and the top plate is moving towards the right with a velocity u_1. The gap between the plates is small and the flow is fully developed.

$\rightarrow Steady, \rightarrow Incompressible, \rightarrow 2-D$

In fully developed flow, v=0

x-momentum $0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2}$

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Ex. $p_1 = p_2$ $\quad \frac{dp}{dx} = \frac{P_2 - P_1}{L} = 0 \rightarrow pure$ Couette flow

$$\frac{d^2u}{dy^2} = 0$$
$$\Rightarrow \frac{du}{dy} = C_1$$
$$\Rightarrow u = C_1 y + C_2$$

B.C.s (1) At y=0, u = 0, (2) At y=H, $u=u_1$

(1) $C_2=0$

(2) $u_1 = C_1 H \Rightarrow C_1 = \frac{u_1}{H}$

$$u = \frac{u_1 y}{H}$$

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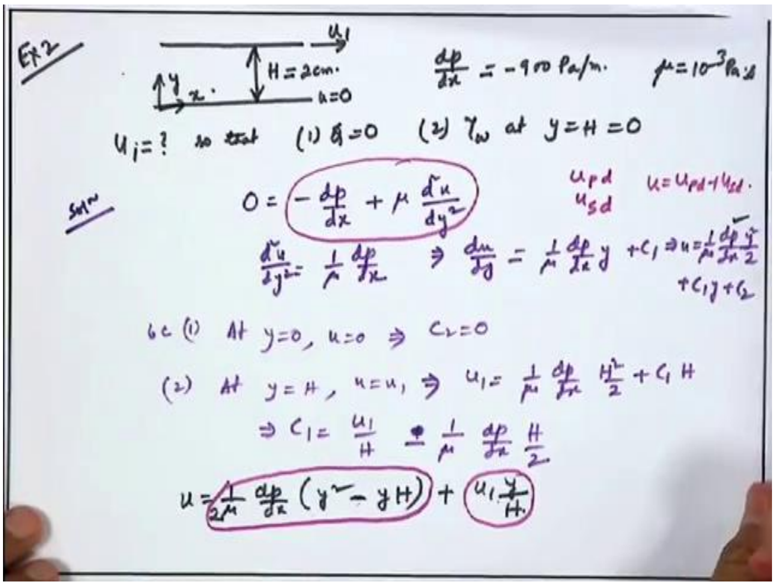

Ex. 2 Consider steady incompressible fully developed flow of water at 20^0 C between two horizontal parallel plates with the gap between the plates is 2 cm. A constant pressure gradient $\frac{dp}{dx} = -900 Pa/m$. The upper plate moves with a uniform speed whereas, the lower plate is stationary. Find the velocity of the upper plate. $\left(\mu = 10^{-3} Pa.s\right)$

Solution:

$$0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2}$$

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \Rightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1 \Rightarrow u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

B.C. (1) At y=0, u =0 $\Rightarrow C_2 = 0$

(2) At y=H, u=u_1 $\Rightarrow u_1 = \frac{1}{\mu}\frac{dp}{dx}\frac{H^2}{2} + C_1 H \Rightarrow C_1 = \frac{u_1}{H} - \frac{1}{\mu}\frac{dp}{dx}\frac{H}{2}$

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yH\right) + u_1\frac{y}{H}$$

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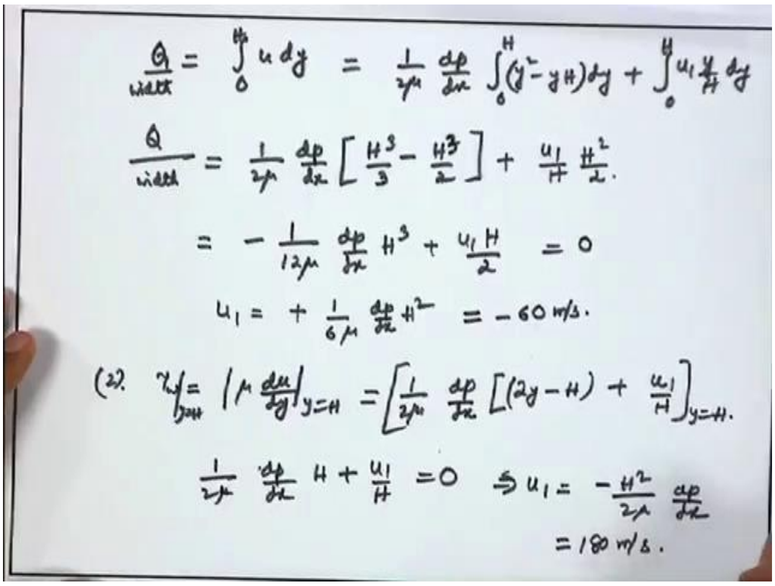

$$\frac{Q}{width} = \int_0^H u dy = \frac{1}{2\mu}\frac{dp}{dx}\int_0^H \left(y^2 - yH\right)dy + \int_0^H u_1 \frac{y}{H}dy$$

$$\frac{Q}{width} = \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{H^3}{3} - \frac{H^2}{2}\right] + \frac{u_1}{H}\frac{H^2}{2} = -\frac{1}{12\mu}\frac{dp}{dx}H^3 + \frac{u_1 H}{2} = 0$$

$$u_1 = +\frac{1}{6\mu}\frac{dp}{dx}H^2 = -60 m/s$$

$$(2)\ \ \tau_w\Big|_{y=H} = \left|\mu\frac{du}{dy}\right|_{y=H} = \frac{1}{2\mu}\frac{dp}{dx}\left[\left(2y - H\right) + \frac{u_1}{H}\right]_{y=H}$$

$$\frac{1}{2\mu}\frac{dp}{dx}H + \frac{u_1}{H} = 0 \Rightarrow u_1 = -\frac{H^2}{2\mu}\frac{dp}{dx} = 180 m/s$$

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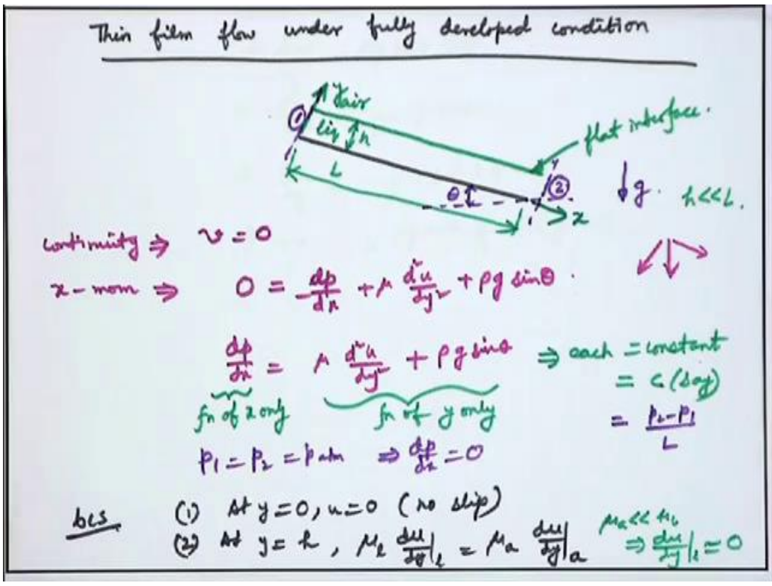

Thin film flow under fully developed condition

$$Continuity \Rightarrow v = 0$$

$$x - momentum \Rightarrow 0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2} + \rho g \sin\theta$$

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2} + \rho g \sin\theta \Rightarrow each = const = C(say) = \frac{P_2 - P_1}{L}$$

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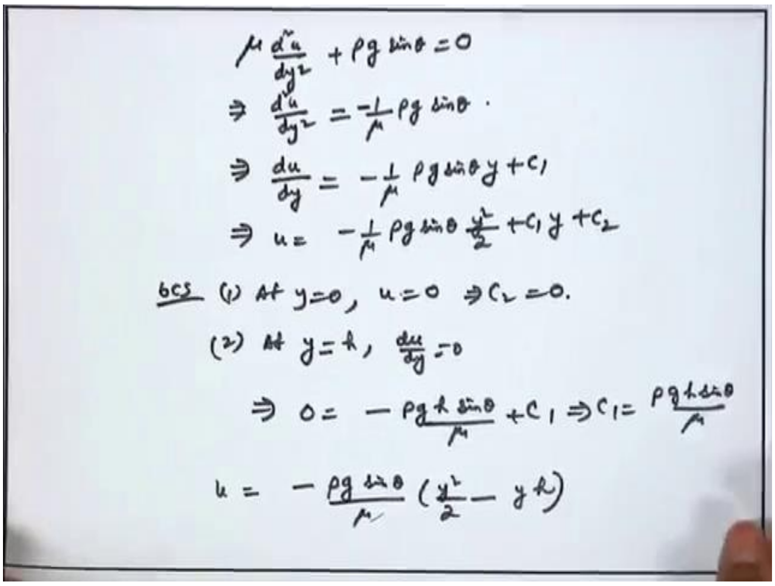

$$\mu \frac{d^2u}{dy^2} + \rho g \sin\theta = 0$$

$$\Rightarrow \frac{d^2u}{dy^2} = -\frac{1}{\mu}\rho g \sin\theta$$

$$\Rightarrow \frac{du}{dy} = -\frac{1}{\mu}\rho g \sin\theta y + C_1$$

$$\Rightarrow u = -\frac{1}{\mu}\rho g \sin\theta \frac{y^2}{2} + C_1 y + C_2$$

B.C.s (1) At y=0, u =0(no slip)

(2) At y=h, $\mu_l \left.\frac{du}{dy}\right|_l = \mu_a \left.\frac{du}{dy}\right|_a$

$$\mu_a << \mu_l \Rightarrow \left.\frac{du}{dy}\right|_l = 0$$

B.C.s (1) At y=0, u =0 $\Rightarrow C_2 = 0$

(2) At y=h, $\frac{du}{dy} = 0$

$$\Rightarrow 0 = -\frac{\rho g h \sin\theta}{\mu} + C_1 \Rightarrow C_1 = \frac{\rho g \sin\theta}{\mu}$$

$$u = -\frac{\rho g \sin\theta}{\mu}\left(\frac{y^2}{2} - yh\right)$$

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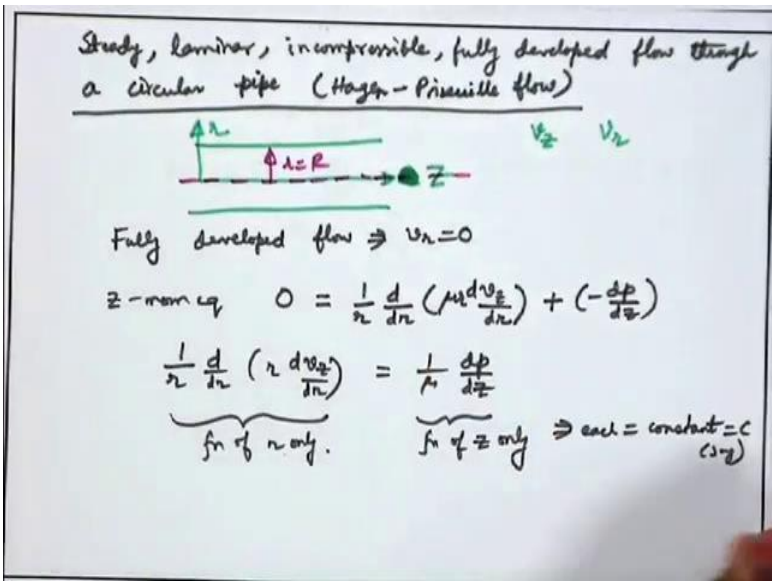

So, we will consider steady, laminar incompressible, fully developed, flow through a circular pipe, this is called as Hagen Poiseuille flow.

Fully developed flow, $v_r=0$

Z-momentum eqn $\quad 0=\frac{1}{r}\frac{d}{dr}\left(\mu r\frac{dv_z}{dr}\right)+\left(-\frac{dp}{dz}\right)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{1}{\mu}\frac{dp}{dz}$$

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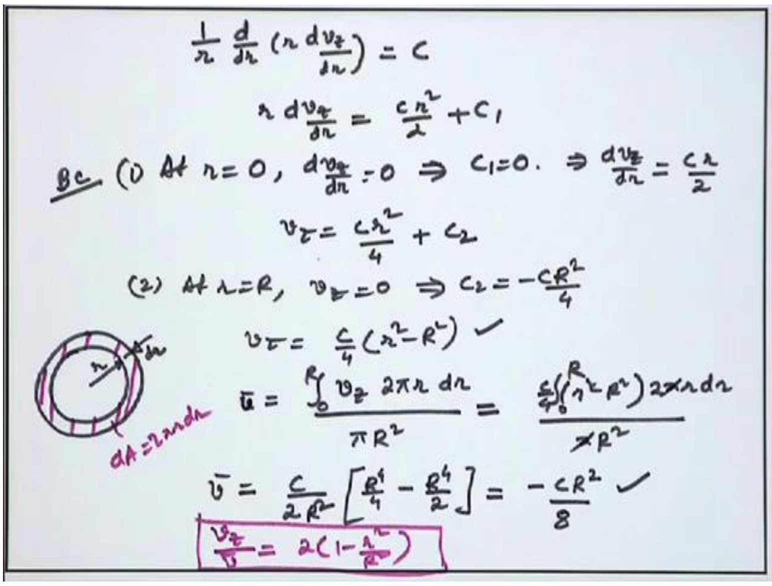

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=C$$

$$r\frac{dv_z}{dr}=\frac{cr^2}{2}+C_1$$

At r=0, $\frac{dv_z}{dr}=0\Rightarrow C_1=0\Rightarrow\frac{dv_z}{dr}=\frac{cr}{2}$

$$v_z=\frac{Cr^2}{2}+C_2$$

At r=R, $v_z=0\Rightarrow C_2=-\frac{CR^2}{4}$

$$v_z=\frac{C}{4}\left(r^2-R^2\right)$$

$$\bar{u}=\frac{\int_0^R v_z 2\pi r dr}{\pi R^2}=\frac{\frac{C}{4}\int_0^R\left(r^2-R^2\right)2\pi r dr}{\pi R^2}$$

$$\bar{v}=\frac{C}{2R^2}\left(\frac{r^4}{4}-\frac{R^4}{2}\right)=-\frac{CR^2}{8}$$

$$\frac{v_z}{\bar{v}}=2\left(1-\frac{r^2}{R^2}\right)$$

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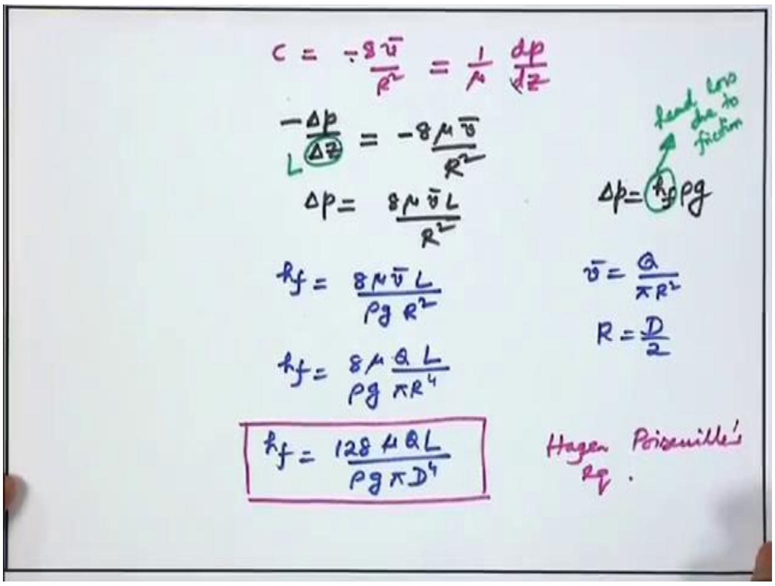

$$C = -8\frac{\bar{v}}{R^2} = \frac{1}{\mu}\frac{dp}{dz}$$

$$\frac{\Delta p}{\Delta z} = -\frac{8\mu\bar{v}}{R^2}$$

$$\Delta p = \frac{8\mu\bar{v}L}{R^2} \qquad \Delta p = h_f \rho g$$

$$h_f = \frac{8\mu\bar{v}L}{\rho g R^2}$$

$$h_f = \frac{8\mu QL}{\rho g \pi R^4}$$

$$h_f = \frac{128\mu QL}{\rho g \pi D^4} \leftarrow \textit{Hagen}\ \text{Poiseuille eqn.}$$

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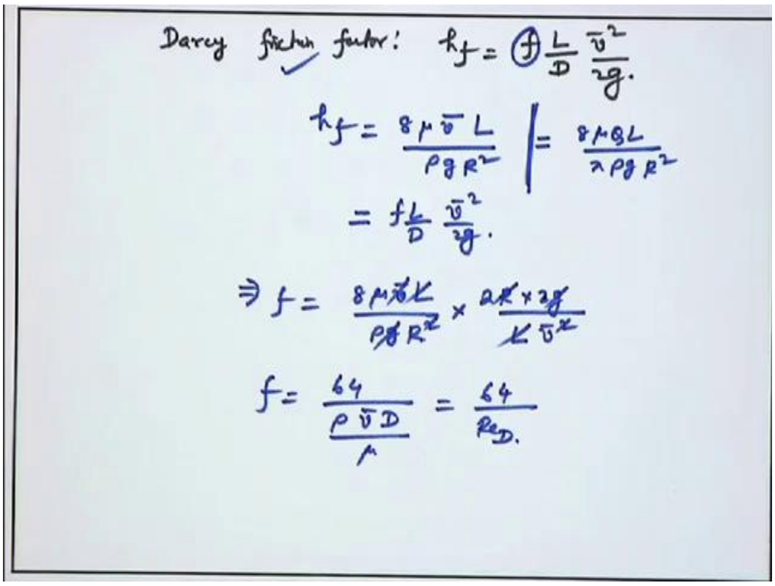

Darcy friction factor: $h_f = f\frac{L}{D}\frac{\bar{v}^2}{2g}$

$$h_f = \frac{8\mu\bar{v}L}{\rho g R^2} = \frac{8\mu QL}{\pi\rho g R^2} = f\frac{L}{D}\frac{\bar{v}^2}{2g}$$

$$f = \frac{8\mu\bar{v}L}{\rho g R^2} \times \frac{2R\times 2g}{L\bar{v}^2}$$

$$f = \frac{64}{\frac{\rho \bar{v} D}{\mu}} = \frac{64}{\mathrm{Re}_D}$$