## **Introduction to Fluid Mechanics Prof. Suman Chakraborty Department of Mechanical Engineering Indian Institute of Technology, Kharagpur**

**Lecture – 54 Navier-Stokes equation- Part – IV**

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So, we will consider a second example of exact solution of the Navier-Stokes equation through something which is very important in fluid mechanics this is called a Couette flow.

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Consider two parallel plates with relative motion; that means, let us say that the bottom plate is stationary and the top plate is moving towards the right with a velocity  $u_1$ . The gap between the plates is small and the flow is fully developed.

 $\rightarrow$  *Steady*,  $\rightarrow$  *Incompressible*,  $\rightarrow$  2 – *D* 

In fully developed flow,  $v=0$ 

x-momentum 2  $0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dx^2}$  $dx \stackrel{\text{f}}{\sim} dy$  $=-\frac{ap}{\mu}+\mu$ 2 2 *d*<sup>2</sup>*u* 1 *dp*  $dy^2$   $\mu$  dx = Ex.  $p_1 = p_2$   $\frac{dp}{dx} = \frac{P_2 - P_1}{L} = 0 \rightarrow pure$  $=\frac{P_2-P_1}{I}=0 \rightarrow pure$  Couette flow 2  $\frac{d^2u}{dx^2} = 0$  $\frac{du}{dt} = C_1$  $\Rightarrow$   $u = C_1 y + C_2$ *dy dy* =  $\Rightarrow \frac{du}{1} = 0$ B.C.s (1) At y=0,  $u = 0$ , (2) At y=H,  $u=u_1$  $(1) C<sub>2</sub>=0$ (2)  $u_1 = C_1 H \implies C_1 = \frac{u_1}{H}$  $u_1 = C_1 H \Rightarrow C_1 = \frac{u}{h}$ *H*  $=C_1 H \Rightarrow C_1 = \frac{u}{h}$  $u = \frac{u_1 y}{f}$ *H* =

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Ex. 2 Consider steady incompressible fully developed flow of water at  $20^{\circ}$  C between two horizontal parallel plates with the gap between the plates is 2 cm. A constant pressure

gradient  $\frac{dp}{dp} = -900 Pa/m$ *dx*  $=$  -900Pa / m. The upper plate moves with a uniform speed whereas, the lower plate is stationary. Find the velocity of the upper plate.  $\mu = 10^{-3} Pa.s$ 

Solution:

$$
0 = -\frac{dp}{dx} + \mu \frac{d^2 u}{dy^2}
$$
  
\n
$$
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} \Rightarrow \frac{du}{dy} = \frac{1}{\mu} \frac{dp}{dx} y + C_1 \Rightarrow u = \frac{1}{\mu} \frac{dp}{dx} \frac{y^2}{2} + C_1 y + C_2
$$

B.C. (1) At y=0,  $u = 0 \implies C_2 = 0$ 

2

(2) At y=H, u=u<sub>1</sub> 
$$
\Rightarrow
$$
  $u_1 = \frac{1}{\mu} \frac{dp}{dx} \frac{H^2}{2} + C_1 H \Rightarrow C_1 = \frac{u_1}{H} - \frac{1}{\mu} \frac{dp}{dx} \frac{H}{2}$   

$$
u = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - yH) + u_1 \frac{y}{H}
$$

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$$
\frac{Q}{u \, du} = \int_{0}^{h} u \, dy = \frac{1}{h} \frac{df}{dx} \int_{0}^{h} (j^{2} - j\, \theta) dy + \int_{0}^{h} u_{1} \frac{dy}{dx} dy
$$
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$$
\frac{Q}{u \, du} = \frac{1}{h} \frac{df}{dx} \left[ \frac{u^{3}}{3} - \frac{u^{3}}{4} \right] + \frac{u_{1}}{4} \frac{u^{3}}{2}.
$$
\n
$$
= -\frac{1}{h^{2}h} \frac{dy}{dx}h^{3} + \frac{u_{1} \frac{1}{h}}{2} = 0
$$
\n
$$
u_{1} = + \frac{1}{h^{2}h} \frac{dy}{dx}h^{2} = -60 \frac{m^{2}h}{2}
$$
\n
$$
\frac{1}{h^{2}} \left[ \frac{2}{3} \frac{dy}{dx} + \frac{u_{1} \frac{1}{h}}{2} \right] = 0 \Rightarrow u_{1} = -\frac{u^{2}}{2h} \frac{dy}{dx}
$$
\n
$$
= 180 \frac{m^{2}h}{2h}
$$

$$
\frac{Q}{width} = \int_{0}^{H} u dy = \frac{1}{2\mu} \frac{dp}{dx} \int_{0}^{H} (y^{2} - yH) dy + \int_{0}^{H} u_{1} \frac{y}{H} dy
$$
  

$$
\frac{Q}{width} = \frac{1}{2\mu} \frac{dp}{dx} \left[ \frac{H^{3}}{3} - \frac{H^{2}}{2} \right] + \frac{u_{1}}{H} \frac{H^{2}}{2} = -\frac{1}{12\mu} \frac{dp}{dx} H^{3} + \frac{u_{1}H}{2} = 0
$$
  

$$
u_{1} = +\frac{1}{6\mu} \frac{dp}{dx} H^{2} = -60m/s
$$
  
(2)  $\tau_{w}|_{y=H} = \left| \mu \frac{du}{dy} \right|_{y=H} = \frac{1}{2\mu} \frac{dp}{dx} \left[ (2y - H) + \frac{u_{1}}{H} \right]_{y=H}$   

$$
\frac{1}{2\mu} \frac{dp}{dt} H + \frac{u_{1}}{dt} = 0 \Rightarrow u_{1} = -\frac{H^{2}}{2} \frac{dp}{dx} = 180m/s
$$

$$
\frac{1}{2\mu} \frac{dp}{dx} H + \frac{u_1}{H} = 0 \Rightarrow u_1 = -\frac{H^2}{2\mu} \frac{dp}{dx} = 180m / s
$$

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## Thin film flow under fully developed condition

$$
\begin{aligned}\n\text{Thinking:} \quad \text{[1]} \quad \text
$$

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$$
\mu \frac{d\mu}{dy} + \rho \frac{\mu \mu \rho = 0}{dy} = \frac{1}{\mu} \rho \frac{\mu \rho \rho = 0}{dy}
$$
\n
$$
\Rightarrow \frac{d\mu}{dy} = -\frac{1}{\mu} \rho \frac{\mu \rho \rho \rho}{dy} + C
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\mu = -\rho \frac{\rho \rho \rho \rho}{dy} (\frac{\mu}{2} - \frac{\nu}{2} + \frac{\rho}{2})
$$

$$
\mu \frac{d^2 u}{dy^2} + \rho g \sin \theta = 0
$$
  
\n
$$
\Rightarrow \frac{d^2 u}{dy^2} = -\frac{1}{\mu} \rho g \sin \theta
$$
  
\n
$$
\Rightarrow \frac{du}{dy} = -\frac{1}{\mu} \rho g \sin \theta y + C_1
$$
  
\n
$$
\Rightarrow u = -\frac{1}{\mu} \rho g \sin \theta \frac{y^2}{2} + C_1 y + C_2
$$

B.C.s (1) At  $y=0$ ,  $u = 0$ (no slip)

(2) At y=h, 
$$
\mu_l \frac{du}{dy}\bigg|_l = \mu_a \frac{du}{dy}\bigg|_a
$$

$$
\mu_a \ll \mu_l \Rightarrow \frac{du}{dy}\bigg|_l = 0
$$

B.C.s (1) At y=0, u =0  $\Rightarrow$  C<sub>2</sub> = 0

(2) At y=h, 
$$
\frac{du}{dy} = 0
$$

$$
\Rightarrow 0 = -\frac{\rho g h \sin \theta}{\mu} + C_1 \Rightarrow C_1 = \frac{\rho g \sin \theta}{\mu}
$$

$$
u = -\frac{\rho g \sin \theta}{\mu} \left( \frac{y^2}{2} - yh \right)
$$

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So, we will consider steady, laminar incompressible, fully developed, flow through a circular pipe, this is called as Hagen Poiseuille flow.

Fully developed flow,  $v_r = 0$ 

Z-momentum eqn  $0 = \frac{1}{r} \frac{d}{dr} \left( \mu r \frac{dv_z}{dr} \right) + \left( -\frac{dp}{dz} \right)$ 

$$
\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}
$$

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$$
\frac{1}{2}x \frac{d}{dx}(x \frac{dy}{dx}) = C
$$
\n
$$
x \frac{dy}{dx} = \frac{c_1x}{x} + C_1
$$
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$$
y \frac{dy}{dx} = 0, \quad d \frac{dy}{dx} = 0 \Rightarrow C_1 = 0. \Rightarrow d \frac{dy}{dx} = \frac{C_1x}{x}
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$$
y \frac{1}{x} = \frac{C_1x}{x} + C_2
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(2) \quad \text{if } x = \frac{C_1x}{x} + C_2
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(3) \quad \text{if } x = \frac{C_1x}{x} + C_2
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y \frac{1}{x} = \frac{C_1x}{x} + C_2
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$$
\frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) = C
$$
\n
$$
r \frac{dv_z}{dr} = \frac{cr^2}{2} + C_1
$$
\nAt r=0,  $\frac{dv_z}{dr} = 0 \Rightarrow C_1 = 0 \Rightarrow \frac{dv_z}{dr} = \frac{cr}{2}$ \n
$$
v_z = \frac{Cr^2}{2} + C_2
$$
\nAt r=R,  $v_z = 0 \Rightarrow C_2 = -\frac{CR^2}{4}$ \n
$$
v_z = \frac{C}{4} \left( r^2 - R^2 \right)
$$
\n
$$
\frac{1}{u} = \frac{\int_0^R v_z 2\pi r dr}{\pi R^2} = \frac{\frac{C}{4} \int_0^R (r^2 - R^2) 2\pi r dr}{\pi R^2}
$$
\n
$$
\overline{v} = \frac{C}{2R^2} \left( \frac{r^4}{4} - \frac{R^4}{2} \right) = -\frac{CR^2}{8}
$$
\n
$$
\frac{v_z}{\overline{v}} = 2 \left( 1 - \frac{r^2}{R^2} \right)
$$

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$$
c = -\frac{3\overline{v}}{r} = \frac{1}{r} + \frac{4r}{r} =
$$
  

$$
-\frac{dp}{r} = -8\frac{r}{r} =
$$
  

$$
-4r = -8\frac{r}{r} =
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$$
4r = 8\frac{r}{r} =
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$$
4r = \frac{8r\overline{v}L}{r^3}
$$
  

$$
4r = \frac{8r\overline{v}L}{r^3}
$$
  

$$
r = \frac{a}{r}
$$

$$
C = -8 \frac{\overline{v}}{R^2} = \frac{1}{\mu} \frac{dp}{dz}
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$$
\frac{\Delta p}{\Delta z} = -\frac{8 \mu \overline{v}}{R^2}
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$$
\Delta p = \frac{8 \mu \overline{v}L}{R^2}
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$$
h_f = \frac{8 \mu \overline{v}L}{\rho g R^2}
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$$
h_f = \frac{8 \mu QL}{\rho g \pi R^4}
$$

4 128  $h_f = \frac{128 \mu Q L}{\sigma g \sigma R^4} \leftarrow Hagen$ <u>ι 28μQι</u><br>ρgπD<sup>'</sup>  $=\frac{128\mu QL}{R^4}$   $\leftarrow$  *Hagen* Poiseuille eqn.

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During fields, find:

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$$
f_{\pm} = \frac{g_{\mu}g_{\mu}}{\rho_{\theta}g_{\mu}} = \frac{g_{\mu}g_{\mu}}{\rho_{\theta}g_{\mu}}
$$
\n
$$
= f_{\frac{\mu}{D}} \frac{\vec{v}_{\mu}}{\vec{v}_{\theta}}
$$
\n
$$
\Rightarrow f = \frac{g_{\mu}\vec{v}_{\mu}}{\rho_{\theta}g_{\mu}} \times \frac{a_{\mu}\vec{v}_{\mu}}{a_{\mu}\vec{v}_{\theta}}
$$
\n
$$
f = \frac{44}{\rho \vec{v}_{\mu}}
$$

Darcy friction factor: 2  $h_f = f \frac{L}{D} \frac{\overline{v}}{2}$ *D g* =

$$
h_f = \frac{8\mu \overline{v}L}{\rho g R^2} = \frac{8\mu QL}{\pi \rho g R^2} = f \frac{L}{D} \frac{\overline{v}^2}{2g}
$$

$$
f = \frac{8\mu \overline{v}L}{\rho g R^2} \times \frac{2R \times 2g}{L \overline{v}^2}
$$

$$
f = \frac{64}{\frac{\rho \overline{v}D}{\mu}} = \frac{64}{\text{Re}_D}
$$