**Introduction to Fluid Mechanics**
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**Lecture – 53**
**Navier-Stokes equation- Part-III**

We have discussed in our previous lecture the governing equation for viscous flow for Newtonian, Stokesian homogeneous and isotropic fluid. And, the corresponding equation known as Navier-Stokes equation let me write that once more.

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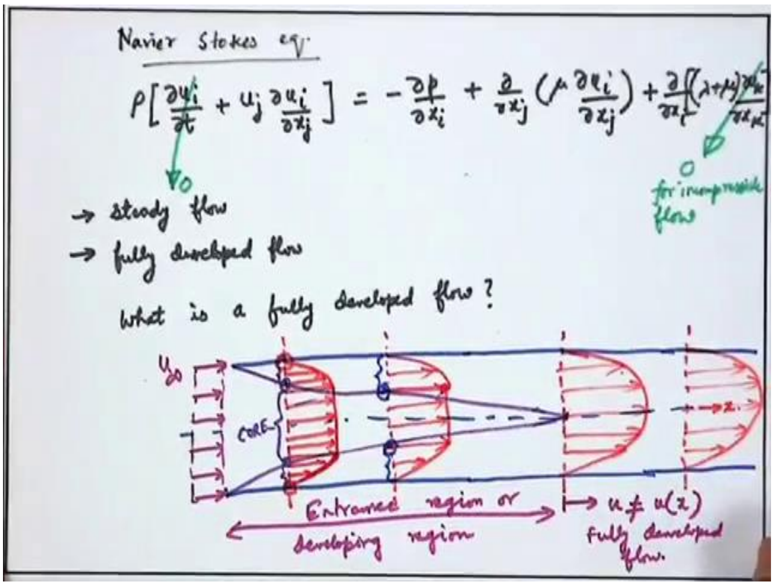


$$\rho\left[\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)+\frac{\partial}{\partial x_i}\left((\lambda+\mu)\frac{\partial u_k}{\partial x_k}\right)+\rho b_i$$

So, let us try to understand what is a fully developed flow. Let us say that we have a channel made of two parallel plates. This is a simplified version of a rectangular channel where there are other boundaries, but for simplicity let us consider a channel with two parallel plates.

Now, let us say that fluid comes from far stream with a velocity $u_\infty$ . Now, this fluid flow will be resisted as the fluid encounters the solid boundary. So, first this velocity profile is uniform; next let us consider another section. Let us say this pipe is quite long so that we can get a fair understanding of what happens as we proceed along the axial direction of the pipe. Axial direction of the pipe is x.

Now, if we draw a velocity profile here what happens at the wall the velocity is 0, because of the no slip boundary condition. Now, further away from the wall the velocity increases.

So, the momentum by the disturbance created by the solid boundary is such that the fluid comes to zero velocity, when it is in contact with the solid boundary and then what happens is that the velocity increases as we go away from the solid boundary then the velocity is uniform and after this again the velocity decreases like this.

Boundary layer region is adjacent to the solid boundary and outside the solid boundary it is called as the core region.

Then, let us go to another section which is further away from the solid boundary further away from the entrance and let us try to draw the velocity profile. As this fluid is now more and more into the channel more and more fluid will be influenced by the effect of the wall. So, the boundary layer will be thicker.

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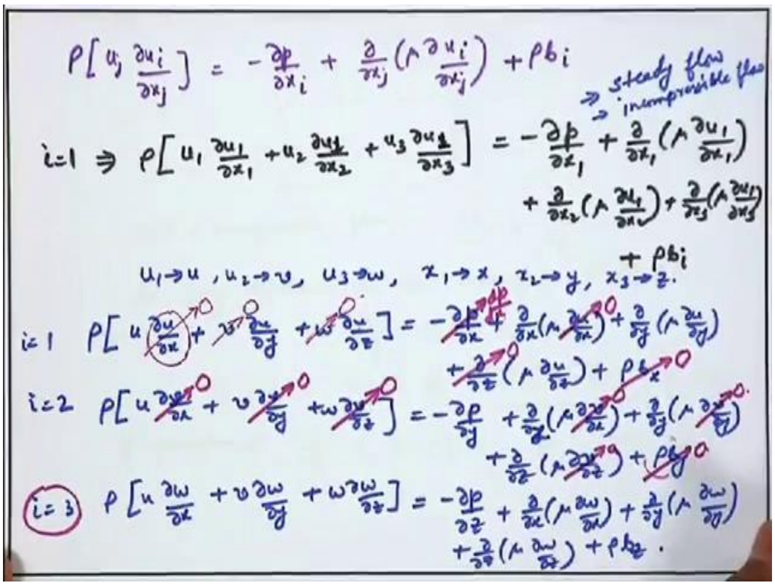


$$\rho\left[u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial \rho}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_i}{\partial x_j}\right) + \rho b_i$$

$$i = 1 \Rightarrow \rho\left[u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_1}\left(\mu \frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\mu \frac{\partial u_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\mu \frac{\partial u_3}{\partial x_3}\right) + \rho b_i$$

Assumptions: 1) Steady flow; 2) Incompressible flow

$$i=1 \Rightarrow \rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+\rho b_x$$

$$i=2 \Rightarrow \rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+\rho b_y$$

$$i=3 \Rightarrow \rho\left[u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)+\rho b_z$$

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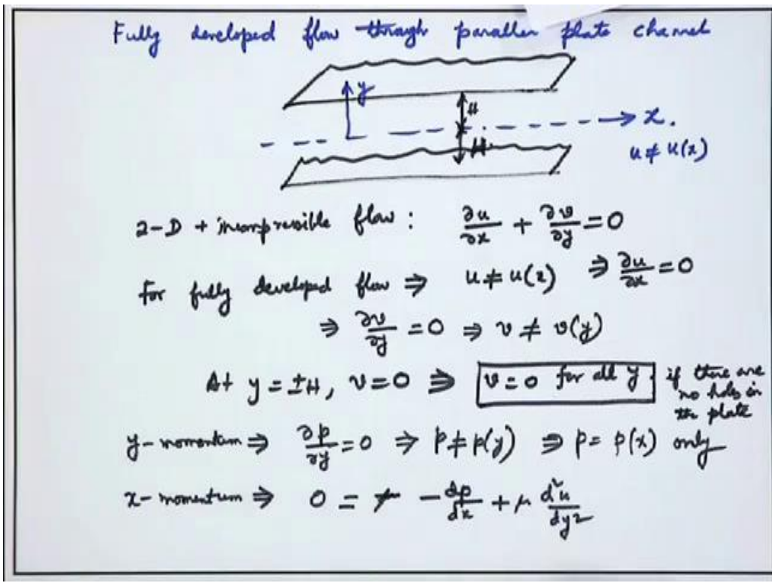


Fully developed flow through parallel plate channel

Let us consider two parallel plates making a parallel plate channel and let us consider that the centre line coordinate is the x and cross coordinate is y.

So, if we are interested to consider fully develop flow we will consider beyond this regime. So, we are not considering what is happening at the entrance region of the pipe or the channel. So, now, for fully developed flow we are writing u is not a function of x, $u \neq u(x)$

2-D incompressible flow: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

For fully developed flow: $u \neq u(x) \Rightarrow \frac{\partial u}{\partial x}=0$

$$\Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v \neq v(y)$$

At y=$\pm H, v = 0 \Rightarrow v = 0$ for all y if there are no holes in the plate

y-momentum $\Rightarrow \frac{\partial p}{\partial y} = 0 \Rightarrow p \neq p(y) \Rightarrow p = p(x)$ only

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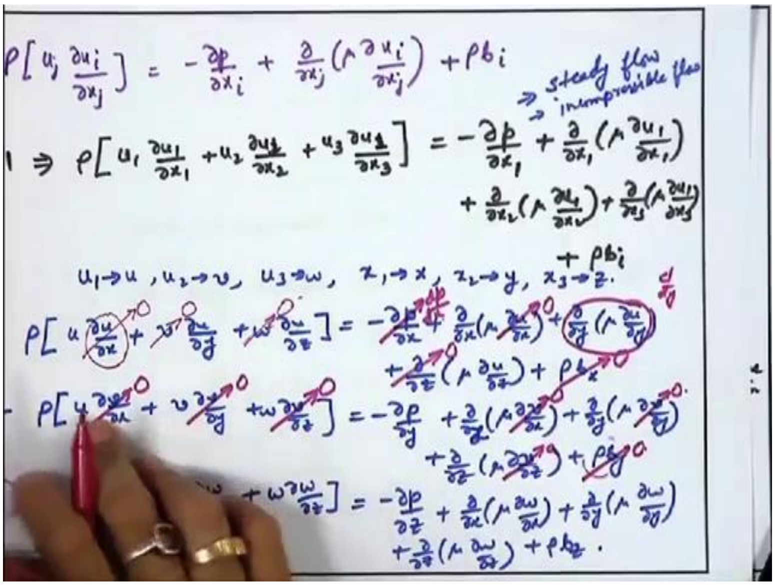


x-momentum$\Rightarrow 0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2}$

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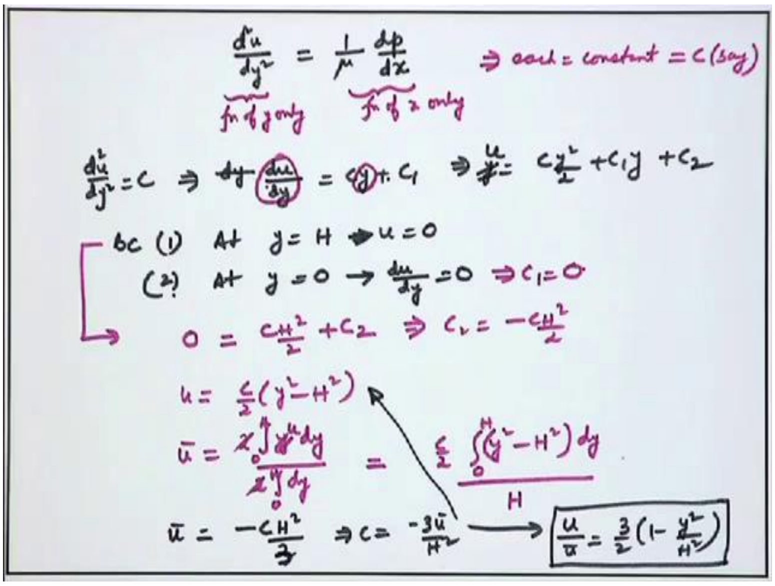

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

$$\frac{d^2u}{dy^2} = c \Rightarrow \frac{du}{dy} = cy + C_1 \Rightarrow y = \frac{cy^2}{2} + C_1 y + C_2$$

B.C.

1) At y=H $\Rightarrow u = 0$

2) At y = 0 $\rightarrow \frac{du}{dy} = 0 \Rightarrow C_1 = 0$

$$0 = \frac{CH^2}{2} + C_2 \Rightarrow C_2 = -\frac{CH^2}{2}$$

$$u = \frac{C}{2}\left(y^2 - H^2\right)$$

$$\bar{u} = \frac{2\int_0^H u dy}{2\int_0^H dy} = \frac{C}{2}\frac{\int_0^H \left(y^2 - H^2\right) dy}{H}$$

$$\frac{u}{\bar{u}} = \frac{3}{2}\left(1 - \frac{y^2}{H^2}\right)$$

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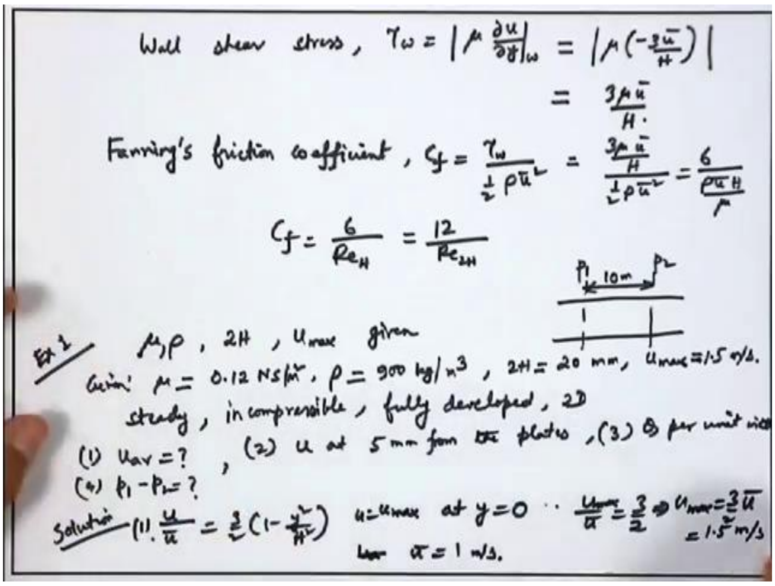


Wall shear stress, $\tau_w = \left|\mu \frac{\partial u}{\partial y}\right| = \left|\mu\left(-\frac{3\bar{u}}{H}\right)\right| = \frac{3\mu\bar{u}}{H}$

Fanning's friction coefficient, $C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{u}^2} = \frac{\frac{3\mu\bar{u}}{H}}{\frac{1}{2}\rho\bar{u}^2} = \frac{6}{\frac{\rho\bar{u}H}{\mu}}$

$$C_f = \frac{6}{\mathrm{Re}_H} = \frac{12}{\mathrm{Re}_{2H}}$$

Ex. 1

Given $\mu = 0.12 Ns/m^2, \rho = 900 Kg/m^3, 2H = 20mm, u_{max} = 1.5 m/s$

Steady, incompressible, fully developed, 2D

Solution:

(1) $\frac{u}{\bar{u}} = \frac{3}{2}\left(1 - \frac{y^2}{H^2}\right)$

U=$u_{max}$ at y=0

$$\frac{u_{\max}}{\bar{u}} = \frac{3}{2} \Rightarrow u_{\max} = \frac{3}{2}\bar{u}$$

$$\Rightarrow \bar{u} = 1m/s$$

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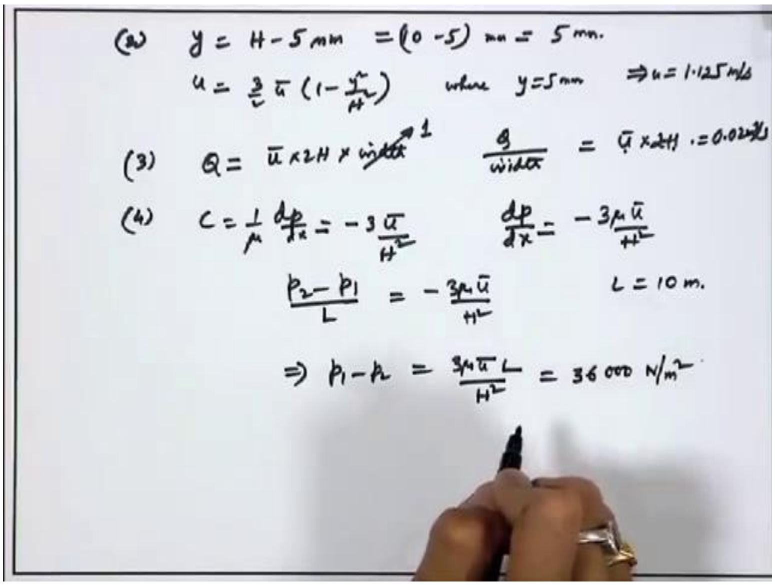


(2) y=H-5=(10-5)mm=5mm

$$\text{u} = \frac{3}{2}\bar{u}\left(1 - \frac{y^2}{H^2}\right) \text{ where y=5mm} \Rightarrow u = 1.125m/s$$

$$(3)\, Q = \bar{u} \times 2H \times 1$$

$$\frac{Q}{width} = \bar{u} \times 2H = 0.02m^3/s$$

$$\text{(4) C} = \frac{1}{\mu}\frac{dp}{du} = -\frac{3\bar{u}}{H^2}$$

$$\frac{dp}{dx} = -\frac{3\mu\bar{u}}{H^2}$$

$$\frac{P_2 - P_1}{L} = -\frac{3\mu\bar{u}}{H^2}$$

$$\Rightarrow P_1 - P_2 = \frac{3\mu\bar{u}L}{H^2} = 36000 N / m^2$$