Introduction to Fluid Mechanics
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Lecture – 51
Navier-Stokes equation- Part-I

Today we will discuss about dynamics of viscous flows that is we will extend those equations for cases in which viscous effects are going to be important. So, it is first important to know that where will the effect of viscous forces come in the governing equations.

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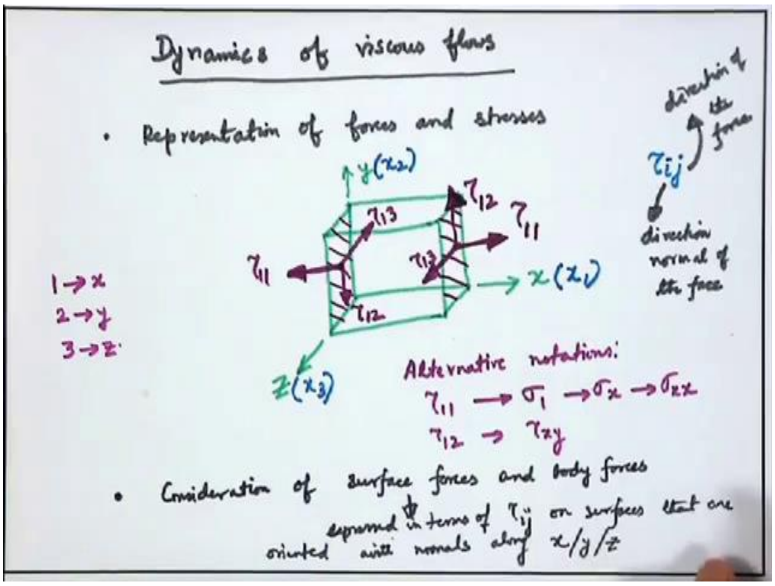

The first agenda is the representation of forces and stresses.

So, representation of forces or stresses to do that; let us consider a rectangular volume of fluid. Let us say this is x direction this is y direction, this is z direction. In an alternative notation this x direction is also called as x_1 direction y direction as x_2 direction and z direction as x_3 direction.

So, the there are 6 faces on this volume element and the speciality of each of these faces is that; each of these faces have normal either along x or along y or along z. Now the forces per unit area on these are represented by the stress tensor components τ_{ij} .

So, i represents the direction normal of the face and j represents the direction of the force. So, let us take an example; so this diagram is just for notation, this diagram is not for calculating the actual stresses and forces and all these; this is just to explain you the notation. So, this face will have force components one normal force component like this.

Alternative notations: $\tau_{11} \rightarrow \sigma_1 \rightarrow \sigma_x \rightarrow \sigma_{xx}$, $\tau_{12} \rightarrow \tau_{xy}$

$1 \rightarrow x, 2 \rightarrow y, 3 \rightarrow z$

So, this τ_{ij} these are not vectors. These are not vectors because vectors will require just one index for specification, but here they require two indices more than one index and the second index is there to describe the direction normal. So, these two indices use of these two indices makes these a second order tensor. Anything with two indices which is not a second may anything merely with two indices is not qualified as a second order tensor it also satisfies some other condition that is it maps a vector onto a vector.

So, we have represented the stresses and the next is consideration of surface forces and body forces.

So, surface forces we have expressed in terms of the stress tensor components and body forces. So, surface forces expressed in terms of τ_{ij} on surfaces that are oriented with normals; along either x or y or z. In addition to the surface forces you also talk about body forces and we have discussed earlier that how to account for the body forces.

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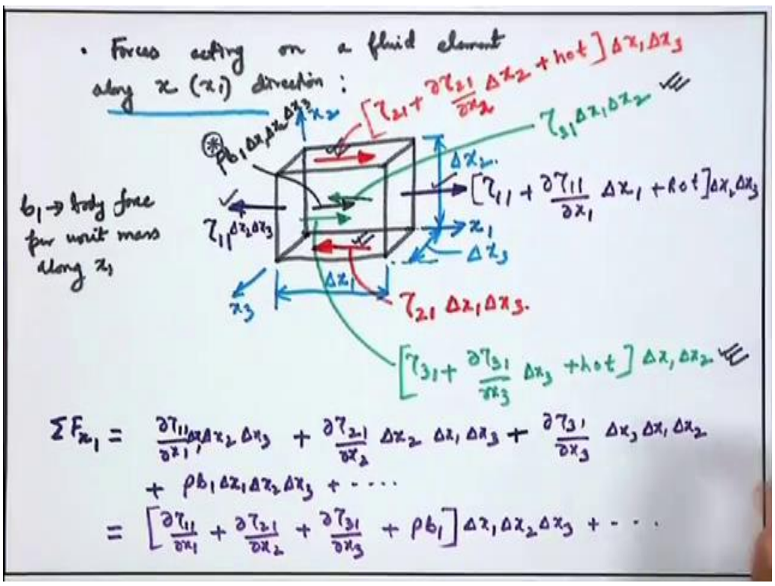

Next we will consider the forces acting on a fluid element; first is x or x_1 direction forces acting on a fluid element along x or x_1 direction. So, let us draw a fluid element this is x_1, this is x_2, this is x_3. So, in this element let us say that this is Δx_1 ; this is Δx_2 and this is Δx_3. So, our job is to represent the forces along the x_1 direction. So, there are 6 faces first we will consider the surface forces there are 6 faces we will consider the surface forces along the x_1 direction.

On the opposite face see now we have to really quantify it on the opposite face here if it is

τ_{11}. On the other side is $\left[\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1}\Delta x_1 + h.o.t\right]$

So, let us say that b_1 is a body force per unit mass along x_1.

$$\sum F_{x_1} = \frac{\partial \tau_{11}}{\partial x_1}\Delta x_1 \Delta x_2 \Delta x_3 + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_2 \Delta x_1 \Delta x_3 + \frac{\partial \tau_{31}}{\partial x_1}\Delta x_3 \Delta x_1 \Delta x_2 + \rho b_1 \Delta x_1 \Delta x_2 \Delta x_3$$
$$= \left[\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{31}}{\partial x_1} + \rho b_1\right]\Delta x_1 \Delta x_2 \Delta x_3 + ...$$

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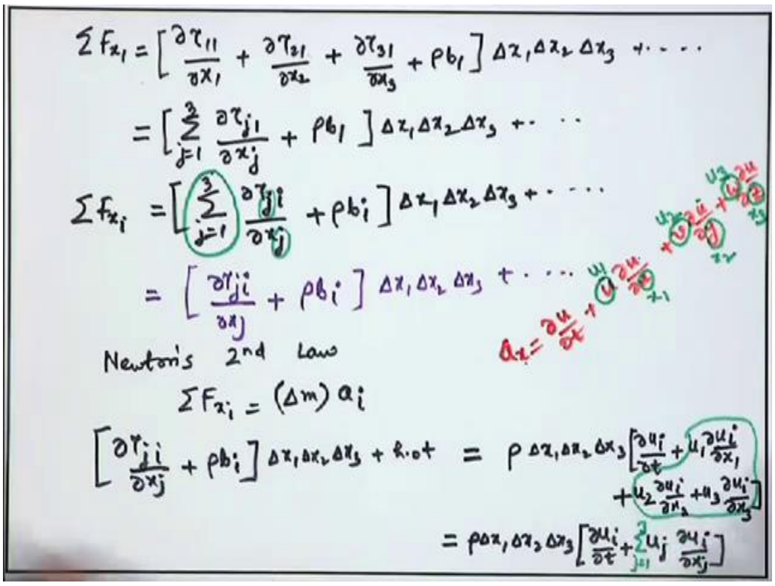

$$\sum F_{x_1} = \left[\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{31}}{\partial x_1} + \rho b_1 \right] \Delta x_1 \Delta x_2 \Delta x_3 + ...$$

$$= \left[\sum_{j=1}^{3} \frac{\partial \tau_{j1}}{\partial x_j} + \rho b_1 \right] \Delta x_1 \Delta x_2 \Delta x_3 + ...$$

$$\sum F_{x_i} = \left[\sum_{j=1}^{3} \frac{\partial \tau_{ji}}{\partial x_j} + \rho b_i \right] \Delta x_1 \Delta x_2 \Delta x_3 + ...$$

$$= \left[\frac{\partial \tau_{ji}}{\partial x_j} + \rho b_i \right] \Delta x_1 \Delta x_2 \Delta x_3 + ...$$

Now we have to remember one thing that in this expression there is a repeated index j and there is a summation over the repeated index. Now Einstein introduced a notation that if there is a repeated index; there must be an invisible summation over the repeated index;

Newton's 2nd Law:

$$\sum F_{x_i} = (\Delta m) a_i$$

$$\left[\frac{\partial \tau_{ji}}{\partial x_j} + \rho b_i \right] \Delta x_1 \Delta x_2 \Delta x_3 + h.o.t = \rho \Delta x_1 \Delta x_2 \Delta x_3 \left[\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} \right]$$

$$= \rho \Delta x_1 \Delta x_2 \Delta x_3 \left[\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_j}{\partial x_j} \right]$$

Acceleration along x , $a_x = \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} + w \dfrac{\partial u}{\partial z}$

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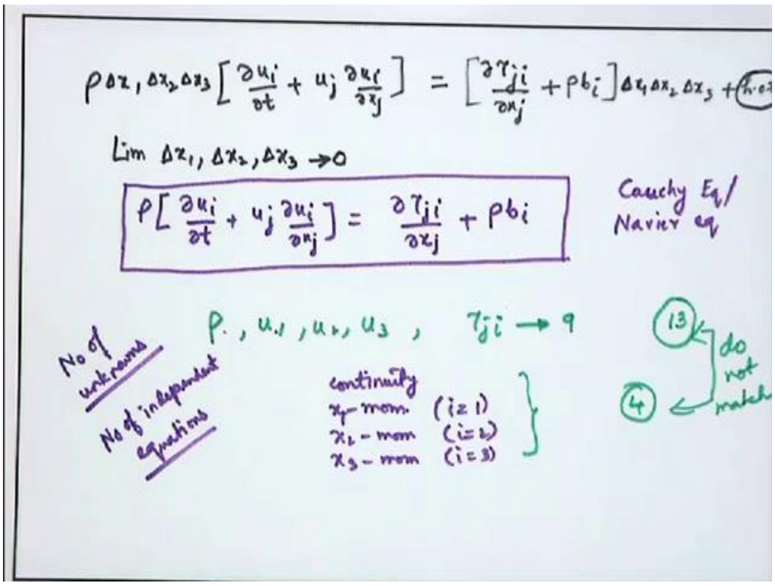

$$\rho \Delta x_1 \Delta x_2 \Delta x_3 \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_j} \right] = \left[\frac{\partial \tau_{ji}}{\partial x_j} + \rho b_i \right] \Delta x_1 \Delta x_2 \Delta x_3 + h.o.t.$$

Lim $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$

$$\rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_j}{\partial x_j} \right] = \left[\frac{\partial \tau_{ji}}{\partial x_j} + \rho b_i \right]$$

So, when we are interested to write the differential equation at a point we will take the limit as $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$. So, that you know the entire volume of fluid shrinks to a point. So, if you take that limit then all the higher order terms will vanish and then we are left with the following equation. This is a very general equation this equation is known as the Cauchy equation or the Navier equation. So, let me write it here; now this equation is very general because we have not assumed any type of fluid.

So, there are total 13 unknowns. Number of independent equations in one equation is the continuity equation which we discussed in the course on kinematics of flow.

So, 4 independent equations and 13 unknowns these do not match; so we require additional equations or additional constraints to close the system. And we our subsequent discussion will be in a quest to get these additional equations so that we can bridge this gap between the number of equations the independent equations and the number of unknown.