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Lecture – 49
Application of RTT: Conservation of angular momentum

We will go ahead with the description of the Reynolds Transport Theorem, in an arbitrarily moving reference frame. So, in the last class we were discussing about the derivative of a vector in a arbitrarily moving reference frame.

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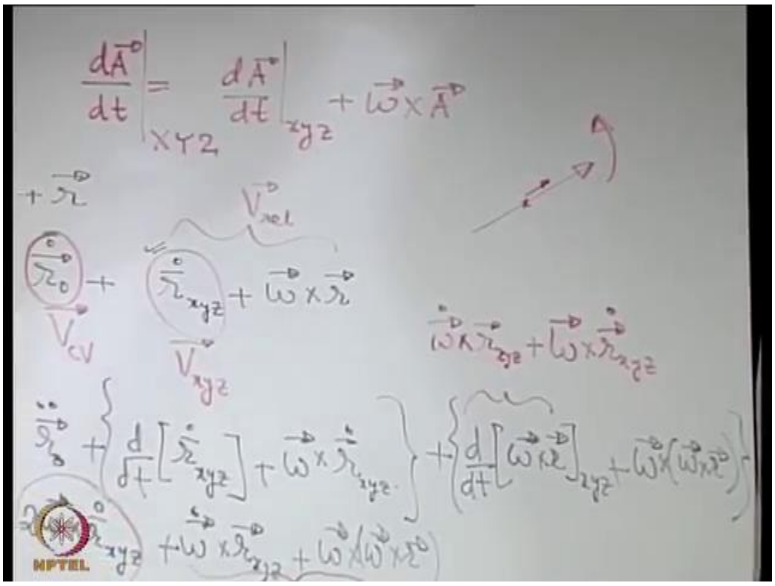

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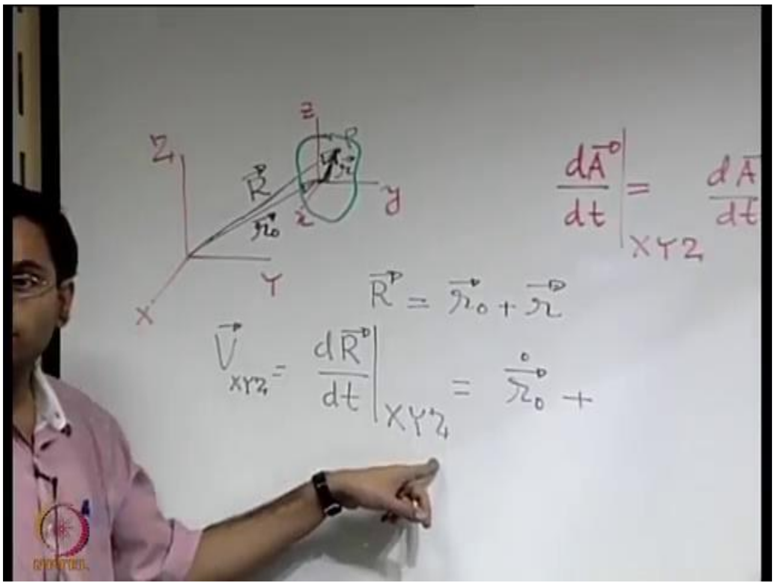

And the result that we saw is if you have two reference frames one is XYZ, another reference frame is xyz where xyz may be having an arbitrary motion, in terms of an angular velocity, in terms of a linear velocity which may even vary with time and so on.

Now, what we will do is we will try to utilize this to figure out that what happens with respect to the velocities and accelerations of fluid elements, located in a control volume because of the movement of the control volume. So, now, we are going to consider, that the control volume is not stationary it is moving. Let us say that we consider the control volume to be such that xyz is attached to the control volume. So, the way in which the control volume moves, xyz motion represents that. So, xyz motion, has two important aspects. One is the translatory motion another is the rotational motion.

If you have the position vector say let us let us call it r_0. The rate of change of these r_0 with respect to time gives the translatory velocity of the xyz reference frame. On the top of that, xyz reference frame is having a rotational velocity in general. There may be special cases when rotational velocity is not there. Now if you consider say a point in the control volume let us say a point P, this point in general represents a point where the fluid has a velocity, acceleration and so on. So, it is a point in the flow field. So, this point, in terms of its position vector is described by this position vector which we say call as R. But if we are trying to analyze everything with respect to the small xyz reference frame, for us the important quantity or the important vector is the position vector of the point P, relative to the origin of the small x y z reference frame.

$$\left.\frac{d\vec{A}}{dt}\right|_{XYZ} = \left.\frac{d\vec{A}}{dt}\right|_{xyz} + \vec{\omega}\times\vec{A}$$

$$\vec{R} = \vec{r}_0 + \vec{r}$$

$$\vec{V}_{XYZ} = \left.\frac{d\vec{R}}{dt}\right|_{XYZ} = \dot{\vec{r}}_0 + \dot{\vec{r}}_{xyz} + \vec{\omega}\times\vec{r}$$

So, these two terms together, may be sort of thought of as a relative velocity. So, this is the absolute velocity, this is the velocity of the control volume this is the velocity relative to the control volume like that. So, the relative velocity has one component because of the translation another component because of the rotation. And if we want to find out the acceleration, acceleration is what is the sole important quantity for us because by that we can relate with the Newton's second law of motion. So, to get acceleration with respect to XYZ, it needs to be differentiated again with respect to time.

$$\dot{\vec{r}}_0 \to \vec{V}_{CV} \ , \ \dot{\vec{r}}_{xyz} \to \vec{V}_{xyz}$$

$$\vec{a}_{XYZ} = \left.\frac{d^2\vec{R}}{dt^2}\right|_{XYZ} = \ddot{\vec{r}}_0 + \left\{\frac{d}{dt}\left[\dot{\vec{r}}_{xyz}\right] + \vec{\omega}\times\dot{\vec{r}}_{xyz}\right\} + \left\{\frac{d}{dt}\left[\vec{w}\times\vec{r}\right]_{xyz} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right\}$$

$$\vec{a}_{XYZ} = \ddot{\vec{r}}_0 + \ddot{\vec{r}}_{xyz} + 2\vec{\omega}\times\dot{\vec{r}}_{xyz} + \dot{\vec{\omega}}\times\vec{r}_{xyz} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

This is because of the linear acceleration of the particle relative to the control volume and this is a combination of the linear and angular motion effect. So, as you all know this is called the Coriolis component of acceleration. So, if you have a reference frame that rotates and relative to that reference frame something translates. So, the combination of that gives rise to an acceleration. So, it the two things which are necessary for this one is the reference frame should be a rotating reference frame, another thing is that there should be a translatory motion relative to the reference film. And then it will experience a sort of acceleration, and that acceleration will try to deflect the particle from its original locus.

And it is very common in particle mechanics also in fluid mechanics just think about ocean currents. So, earth is like a rotating object and on the top of the earth over the earth the ocean currents are moving. So, you have the water moving in the sea with a particular velocity on the top of a reference frame which is rotating. And that is why and the rotational sense is different at different places. So, you will see that there will be a certain deflection of the ocean current in the northern hemisphere and in the southern hemisphere these things you have studied in junior classes of geography. But these are these are very important examples of the implications of Coriolis effects in fluid mechanics. Now what we can do is we can just write it in a bit of a more compact way we can write that.

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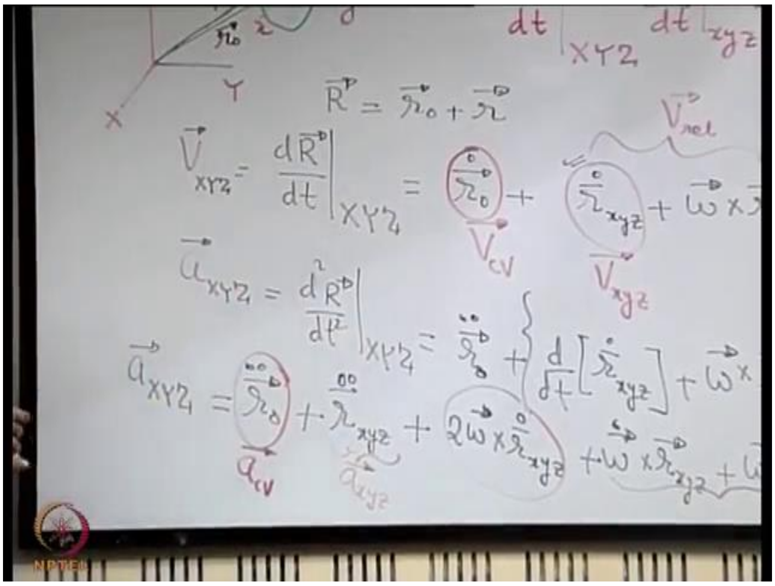

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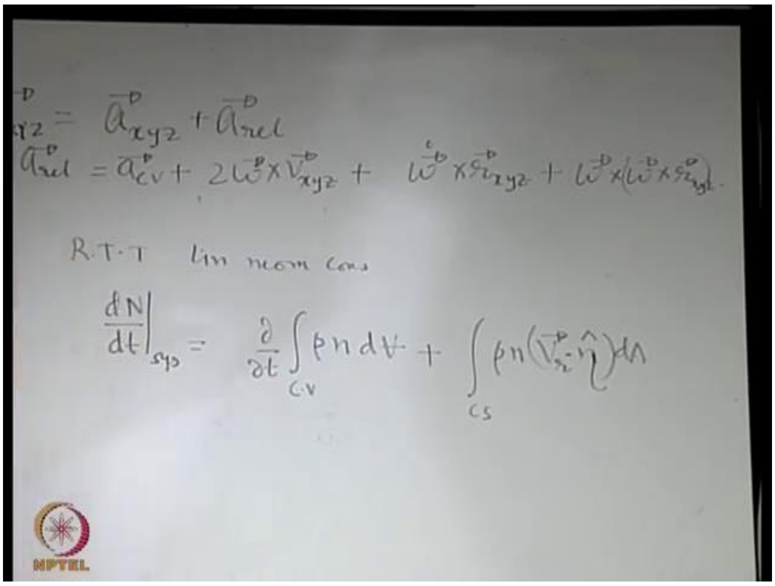

$$\vec{a}_{XYZ} = \vec{a}_{xyz} + \vec{a}_{rel}$$

$$\vec{a}_{rel} = \vec{a}_{CV} + 2\vec{\omega}\times\vec{v}_{xyz} + \dot{\vec{\omega}}\times\vec{r}_{xyz} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{xyz}\right)$$

Reynolds Transport Theorem for Linear Momentum Conservation

$$\left.\frac{dN}{dt}\right|_{system} = \frac{\partial}{\partial t}\int_{CV} \rho n d\forall + \int_{CS} \rho n\left(\vec{V}_r.\hat{\eta}\right) dA$$

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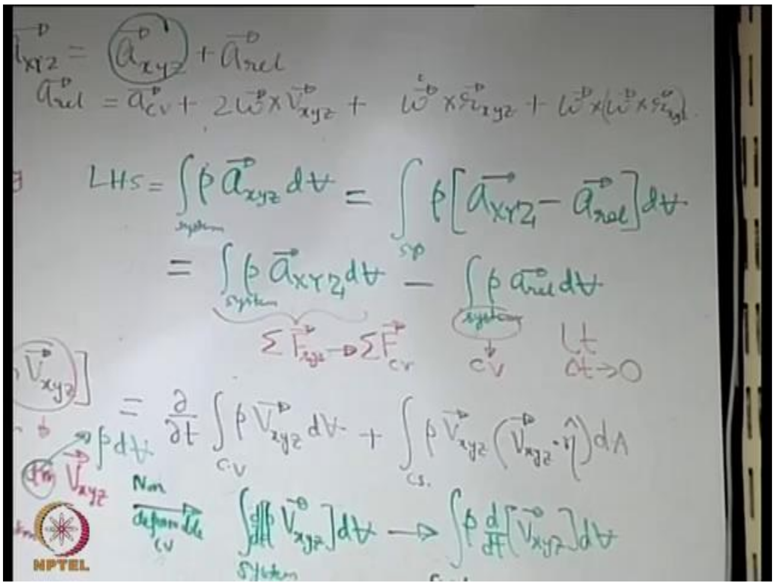

$$LHS = \int \rho \vec{a}_{XYZ} d\forall = \int \rho\left[\vec{a}_{XYZ} - \vec{a}_{rel}\right] d\forall$$
$$= \int \rho \vec{a}_{XYZ} d\forall - \int \rho \vec{a}_{rel} d\forall$$

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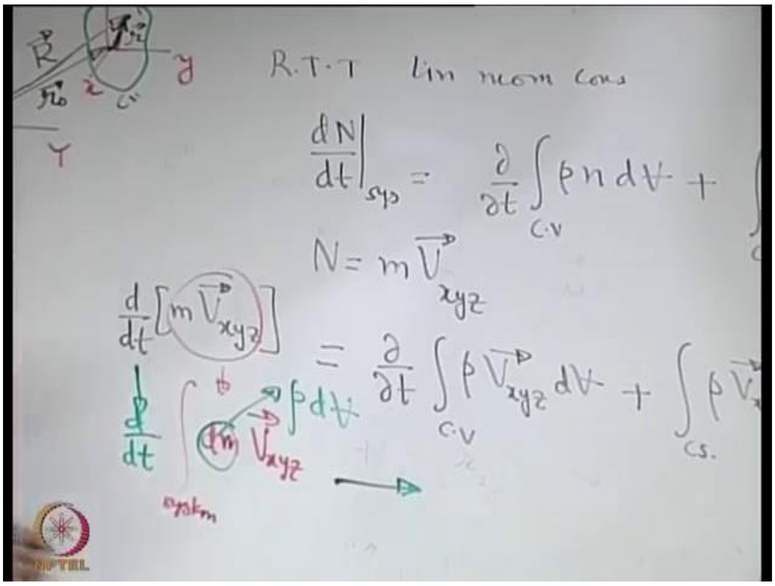

$$N = m\vec{V}_{xyz}$$

$$\frac{d}{dt}\left[m\vec{V}_{xyz}\right] = \frac{\partial}{\partial t}\int_{CV} \rho\vec{V}_{xyz} d\forall + \int_{CS} \rho\vec{V}_{xyz}\left(\vec{V}_{xyz}.\hat{\eta}\right)dA$$

$$\int_{system} \frac{d}{dt}\left[\vec{V}_{xyz}\right] d\forall \rightarrow \int_{system} \rho \frac{d}{dt}\left[\vec{V}_{xyz}\right] d\forall$$

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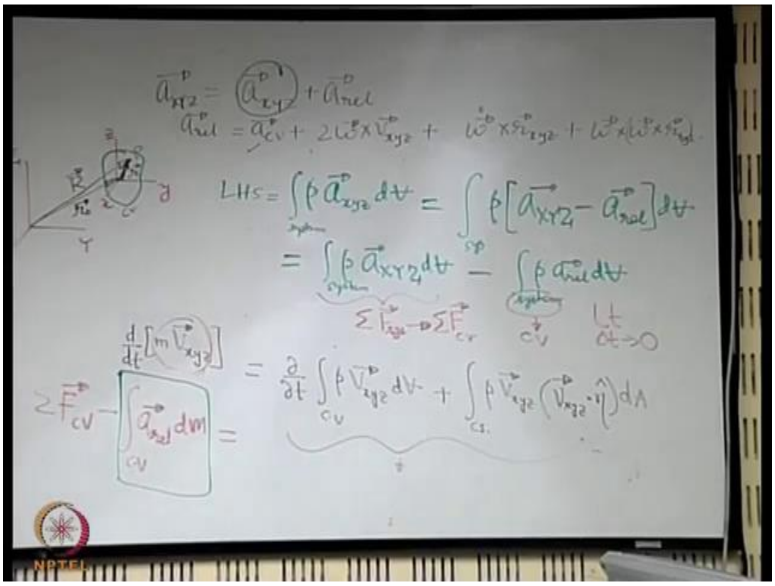

$$LHS = \int_{system} \rho \vec{a}_{xyz} \, d\forall = \int_{sys} \rho \left[\vec{a}_{XYZ} - \vec{a}_{rel} \right] d\forall$$

$$= \int_{system} \rho \vec{a}_{XYZ} \, d\forall - \int_{system} \rho a_{rel} \, d\forall$$

$$\sum \vec{F}_{CV} - \int_{CV} \vec{a}_{rel} \, dm$$