Introduction to Fluid Mechanics
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Lecture –41
Reynolds Transport Theorem (RTT)

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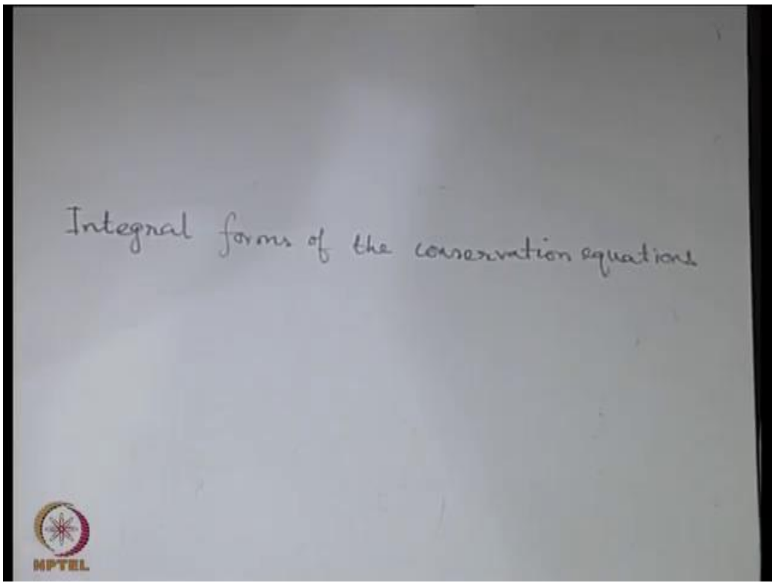

Today we will start with a new chapter which is integral forms of the conservation equations. This is a logical extension of some of our previous discussions where we were discussing about the differential forms of the conservation equations.

And some of the differential forms of the conservation equations were in terms of the conservation of mass; like we discussed about the continuity equation. And conservation of the linear momentum may be expressed to the Newton's second law of motion that was the Euler's equation of motion for an inviscid flow. And subsequently it could give rise to a form of mechanical energy conservation also, but most of those forms that we discussed were of differential in nature.

We will try to look into the feature of integral forms of these conservation equations. But before going into that we must try to develop a feel of what we mean by these conservation equations and how they are important in fluid mechanics. So, when we talk about conservation equations we are talking about certain mathematical form, which represents the physical meaning of conservation of something and we will try to see that what is the basic physical principle that gives rise to the sense of conservation of mass, momentum, energy and how we can express that in an equivalent mathematical form.

When we talk about a conservation principle it is not a bad idea to discuss about it in a bit of a more general framework; that is not very specific to maybe mass or momentum.

Let us say that you have gone to a bank to open a bank account. So, your bank account starts at a given instant where let us say this is like a symbolic representation of your bank account.

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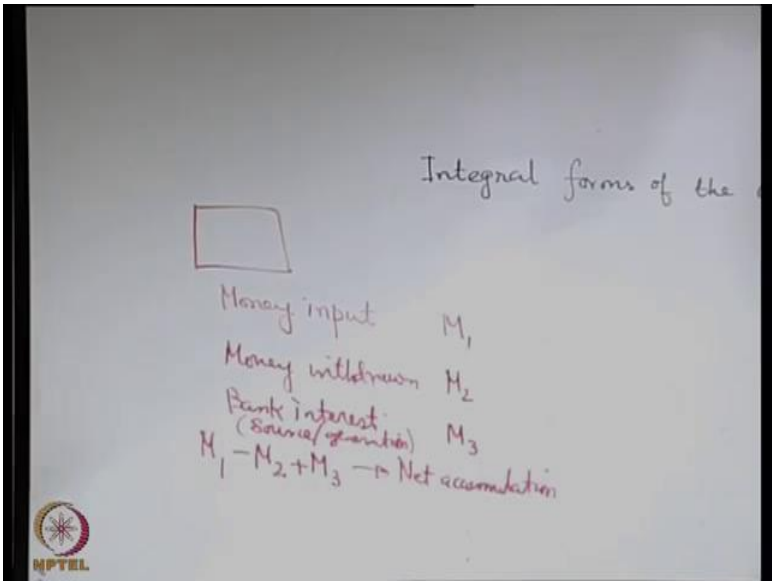

So, you have put some initial money in your bank account and the bank account operates from then onwards. Now say you are working in a job earning a fabulous salary, but you are having such a great amount of money otherwise that you do not care about what is going on in your bank account.

Let accumulated total money amount input is M_1. Money withdrawn if it is M_2 and say the bank interest is M_3.

$M_1 - M_2 + M_3 \rightarrow Net$ accumulation

So, these talks about a principle of conservation of money in a bank account, say the bank account is like a control volume. So, across which there is something which is entering and something which is leaving. We have talked about mass entering momentum entering, but here is like money entering and money leaving.

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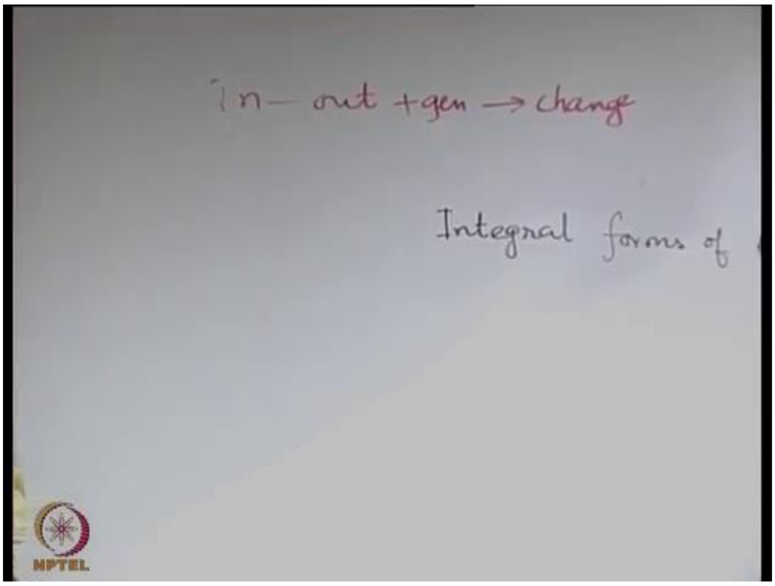

Then something is generated , like interest is leading to a net change. So, this is very simple you cannot even think of a simpler way of looking into a conservation. And believe me or not whatever equations of integral or differential form that we write in fluid mechanics representing conservation of various quantities fundamentally follow this principle.

So, we will try to see that how from this fundamental principle we can develop or we can derive some of the very important conservation equations in fluid mechanics. When we want to do that, we need to keep in mind that there is a general way of looking into this conservation as represented by the simple statement. So, our objective will be to express this simple statement in a somewhat mathematical sense. There is a very important theorem that tells us that how to go about that expression and that theorem is known as Reynolds transport theorem

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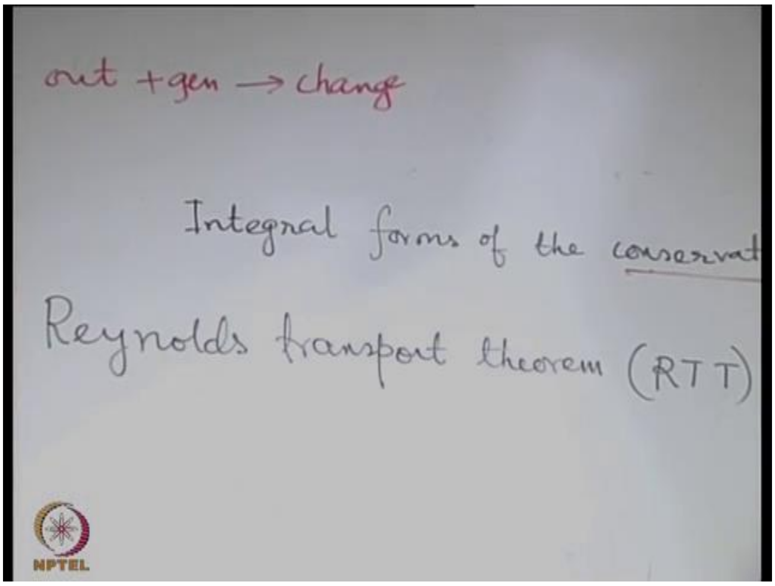

So, one will be tempted to look into the perspective of this theorem that is what this theorem is essentially trying to do. So, before going into the theorem let us try to understand the motivation behind a theorem or in particular the Reynolds transport theorem. We have discussed many times that the difference between a traditional way of looking into mechanics say particle mechanics and fluid mechanics is the reference frame; mostly like in particle mechanics you are looking for a Lagrangian reference frame and in fluid mechanics mostly we are discussing in the context of an Eulerian reference frame.

The reason is obvious that fluids are continuously deforming and it is virtually impossible or very difficult to track individual fluid particles and see that how they are evolving. So, the approach that fits mostly with fluid mechanics is a control volume approach. On the other hand all the basic equations which have been classically developed in mechanics have been not based on a control volume, but based on something which is of a fixed mass and identity and that we call as a system that also we have discussed earlier.

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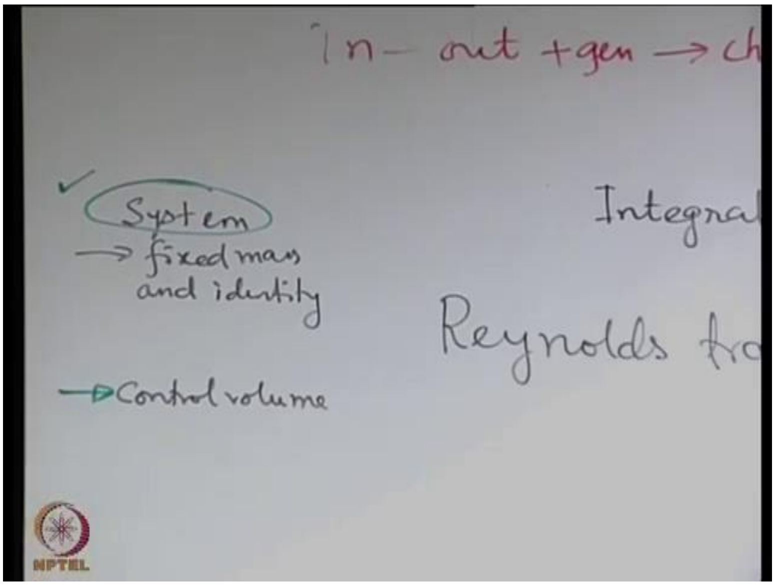

So, we have something as a system which is a fixed mass and identity. And we have something as a control volume which is an identified region in space across which any mass energy or whatever can flow.

So, in very brief transformation from system to control volume or maybe vice versa from control volume to system and that is given by this Reynolds transport theorem. So, the motivation of this is to get a general formulation which will try to give us a guideline of how to have a transformation from a system approach to a control volume approach. So, to do that we will start with like a very simple way of looking into the theorem and maybe we start with a sketch of what happens to a system and what happens to a control volume.

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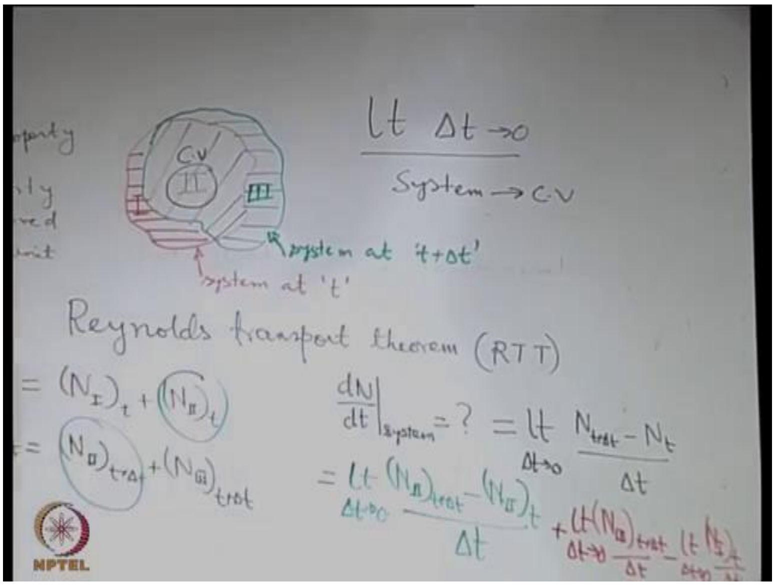

So, let us say that we define a system with its system boundary like this. So, this is an arbitrary way of just sketching a system. There may be some fluid inside which is the boundary and the fluid particles which are there inside are identified. So, that is the system let us say that this is the system boundary at time t.

Now, let us consider a small time interval of Δt; over the small time interval of Δt this system boundary has now occupied a different configuration. In reality if the Δt is small this configuration which is there with the green color is almost merging with that of what was the configuration at time t drawn with the red color. Just for distinguishing these two we have really amplified the change here in the figure, but keep in mind that these two envelopes of the system which are the so called system boundaries will be actually almost merging on one on the top of the other as $\Delta t \to 0$.

So, this we call as system boundary at time $(t + \Delta t)$ keeping in mind Δt is very small. So, this is just to isolate the two system boundaries that we have drawn in a magnified way, but please keep in mind that these two are almost coincident not that they are coincident, but they are virtually falling one on the top of the other. Now when we have drawn a sketch as shown, you can clearly see that there are three important regions in this sketch and one important region straight away is the common intersection between the two configurations.

Let us say the name of the first zone which is masked with the red one is I the next zone is II and the third one that is to the right is III; these are just 3 different volumes that we identify for our own convenience of demarcation. Now what we can say let us say that we want to write a conservation law for some quantities. So, we are not really committing what is the quantity, keep in mind the quantity that we are interested to conserve may be mass, may be momentum, may be energy or whatever, but let us say that we are interested to conserve some quantity.

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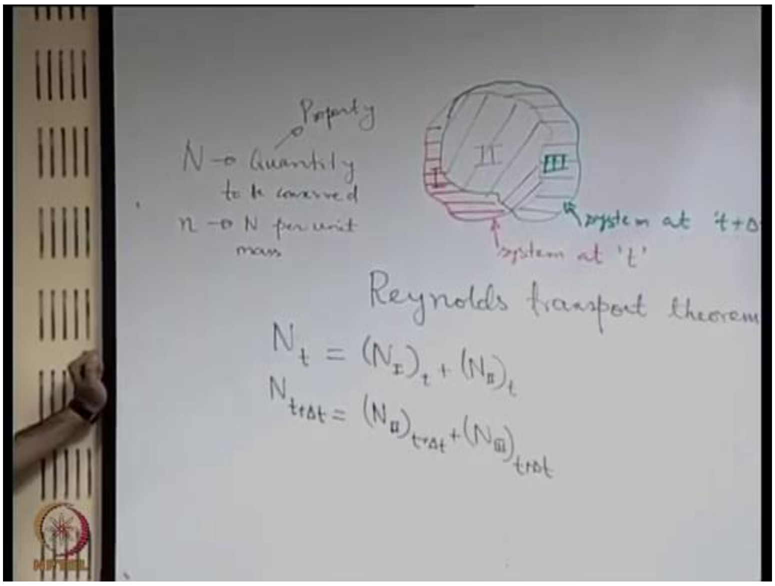

So, let us say that N is the quantity that is conserved; quantity to be conserved means it is satisfying the basic conservation principle that we discussed through an example before this. And let us say that n is N per unit mass. So, N is like it is the total and n is that expressed per unit mass and that total may be any property.

$$N_t = \left(N_I\right)_t + \left(N_{II}\right)_t$$

$$N_{t+\Delta t} = \left(N_{II}\right)_{t+\Delta t} + \left(N_{III}\right)_{t+\Delta t}$$

$$\left.\frac{dN}{dt}\right|_{system} = \underset{\Delta t \to 0}{Lt} \frac{N_{t+\Delta t} - N_t}{\Delta t} = \underset{\Delta t \to 0}{Lt} \frac{(N_{II})_{t+\Delta t} - (N_{II})_t}{\Delta t} + \underset{\Delta t \to 0}{Lt} \frac{\left(N_{III}\right)_{t+\Delta t}}{\Delta t} - \underset{\Delta t \to 0}{Lt} \frac{\left(N_I\right)_t}{\Delta t}$$

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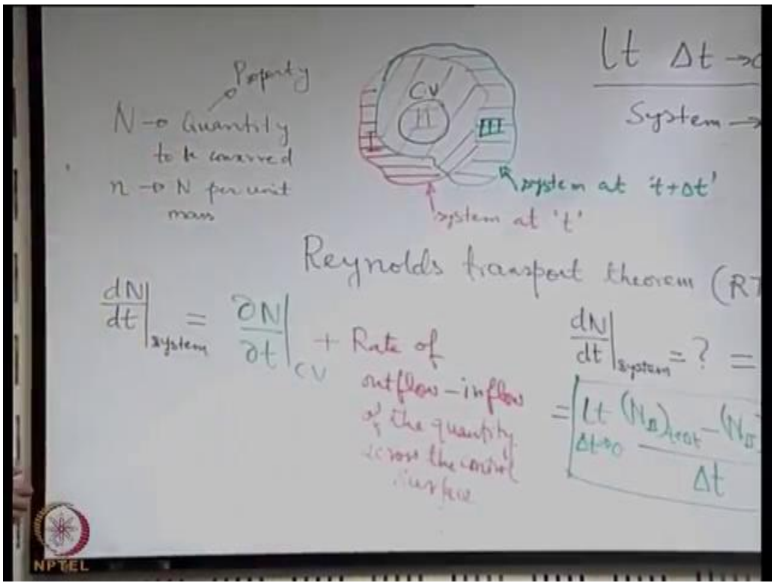

In the limit as $\Delta t \to 0$ say this is a very small region; so, this represents some fluid which is being located on the surface of the control volume. So, in the limit t$\to$ 0, this volume will shrink to almost like located on the surface of the control volume. So, at the surface of the control volume there is some fluid and in the limit as $\Delta t \to 0$; then the term corresponding to I represents a rate process. So, per unit time is a rate process; when the system tends to control volume this volume effect almost tends to a surface effect. So, at the surface of the control volume at the time t; it represents that there is some rate of transport of the rate of influx of the quantity that you are looking for. So, it is like the rate of inflow of the quantity across the surface of the control volume that is called as a control surface; surface of the control volume.

$$\left.\frac{dN}{dt}\right|_{system} = \left.\frac{\partial N}{\partial t}\right|_{CV} \text{+Rate of outflow-inflow of the quantity across the control surface}$$

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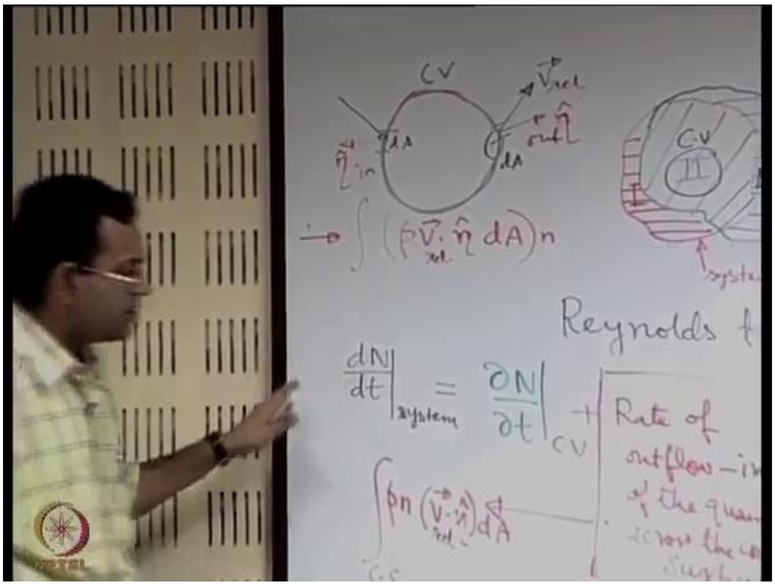

Let us say that this is the control volume and we are interested to develop a mathematical expression for the rate of outflow of the quantity.

So, let us say that you identify a small area on the surface over which the fluid has a velocity V. So, the fluid let us say is going out of the control volume over the surface; it is us maybe it is a part of that surface over which it is leaving. So, it is a part of the outflow boundary.

So, out of the total boundary of the control volume, there may be some part over there is no inflow or outflow maybe it is like a wall; there maybe some part across which fluid is entering or leaving may be its like a hole. So, this is such a place across which the fluid is leaving as an example. Though the fluid is leaving with a particular velocity and it is arbitrary ; the area that is being identified we have seen it many times that area has an important directionality.

So, this area has to be identified with a unit vector $\hat{\eta}$.

$\left(\rho.\vec{V}.\hat{\eta}.dA\right)n$ is the rate of flow of property.

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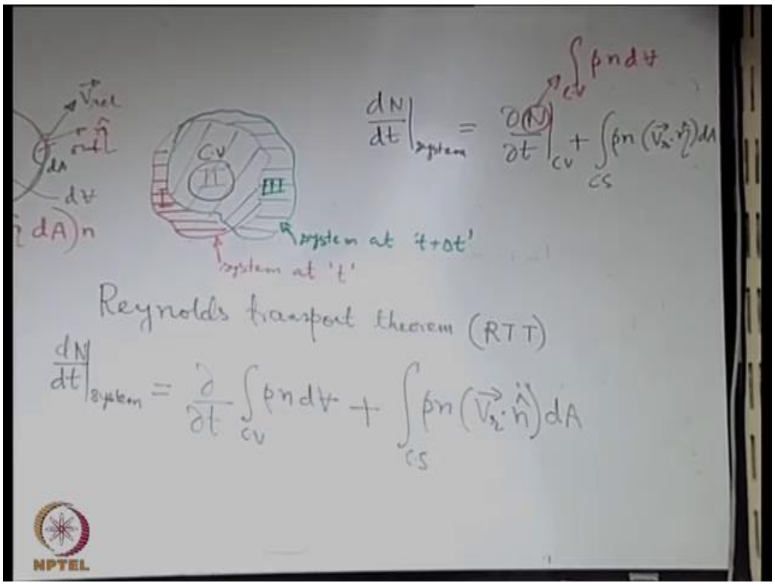

$$\left.\frac{dN}{dt}\right|_{system}=\left.\frac{\partial N}{\partial t}\right|_{CV}+\int_{CS}\rho n\left(\vec{V}_r.\hat{\eta}\right)dA$$

$$N\rightarrow\int_{CV}\rho n d\cancel{V}$$

$$\left.\frac{dN}{dt}\right|_{system}=\frac{\partial}{\partial t}\int_{cv}\rho n d\cancel{V}+\int_{cs}\rho n\left(\vec{V}_r.\hat{n}\right)dA$$