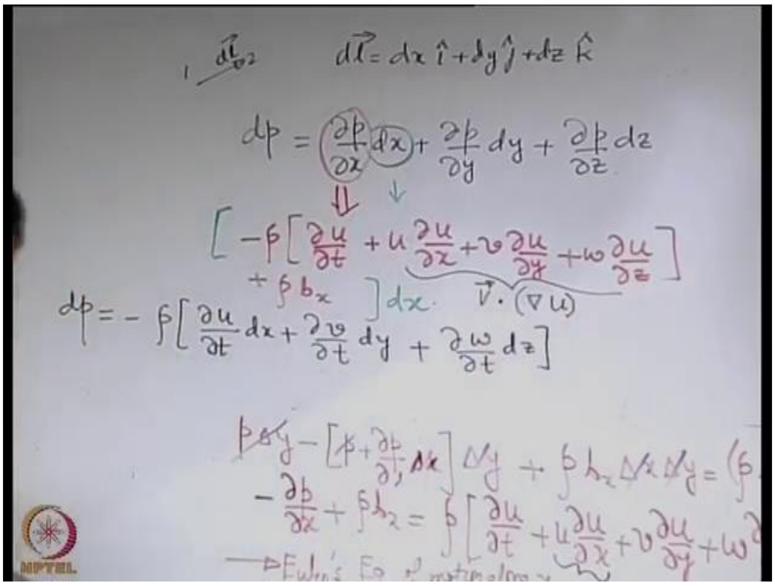
Introduction to Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 32
Bernoulli's equation-Part-I

So, to do that what we will do, we will leave this example, and go back to the Euler's equations of motion along the different directions. So, we have written the Euler's equation of motion along x, which is there in the board. Similar equations are there along y and z. We are interested to write or to find what is the difference in pressure between two points.

(Refer Slide Time: 00:50)



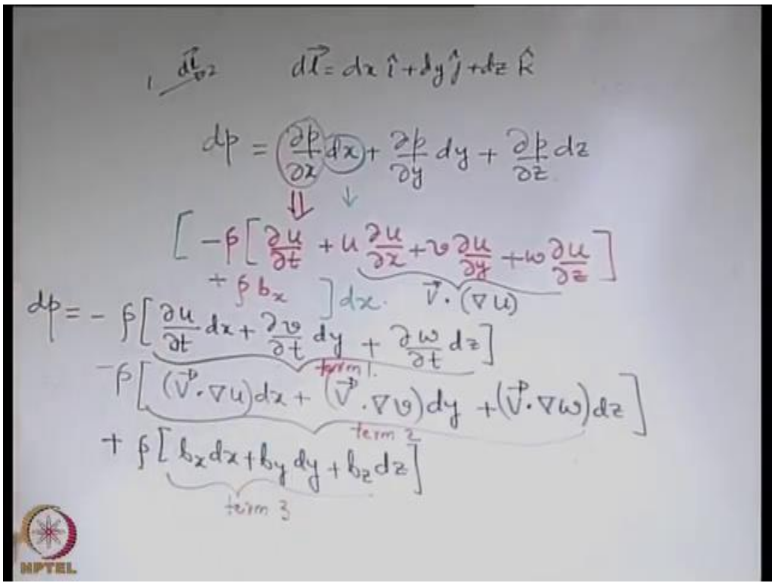
So, let us say that we have two points 1 and 2, which are quite close, so that they are connected by a position vector dl; $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.
To do that, we will note that if you want to find out the difference in pressure between the two points, here pressure is a function of what, x, y, and z. So, you can write this as

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

$$\frac{\partial p}{\partial x} = -\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] + \rho b_x$$

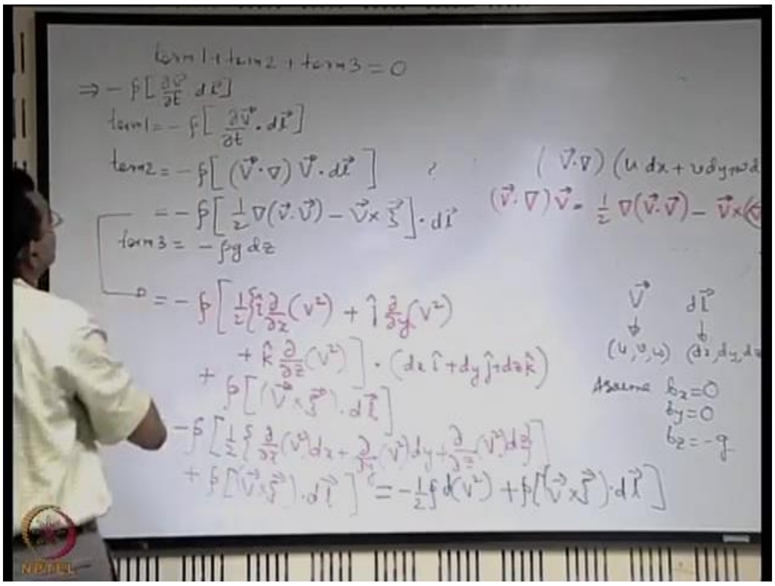
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \vec{V}.(\nabla u)$$

(Refer Slide Time: 05:20)



$$dp = -\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] - \rho\left[\left(\vec{V}.(\nabla u)\right)dx + \left(\vec{V}.(\nabla v)\right)dy + \left(\vec{V}.(\nabla w)\right)dz\right] + \rho\left[b_x dx + b_y dy + b_z dz\right]$$

(Refer Slide Time: 06:49)



$$Term1 = -\rho\left[\frac{\partial \vec{v}}{\partial t}.d\vec{t}\right]$$

$$Term2 = -\rho\left[\left(\vec{v}.\nabla\right)\vec{v}d\vec{l}\right]$$

$$\left(\vec{v}.\nabla\right).\vec{v} = \frac{1}{2}\nabla\left(\vec{v}.\vec{v}\right) - \vec{v}\times(\nabla\times\vec{v})$$

Now, why we are putting in this particular form is because, here you get the vorticity vector. And we were finding out that the condition of rotationality or irrotationality has some influence on the pressure difference between the points. And this vector solely is responsible for whether it is rotational or irrotational. So, we will put that simplification here,

$$Term2 = -\rho\left[\left(\vec{v}.\nabla\right)\vec{v}d\vec{l}\,\right]$$

$$= -\rho\left[\frac{1}{2}\nabla\left(\vec{v}.\vec{v}\right) - \vec{v}\times\vec{\zeta}\,\right].d\vec{l}$$

For the term 3, what we will assume the it is again a very general term. What we will assume that the gravity is the only body force, which acts along the negative z direction as we considered in the problem that we discussed just before this. $b_x=0$, $b_y=0$, $b_z=-g$

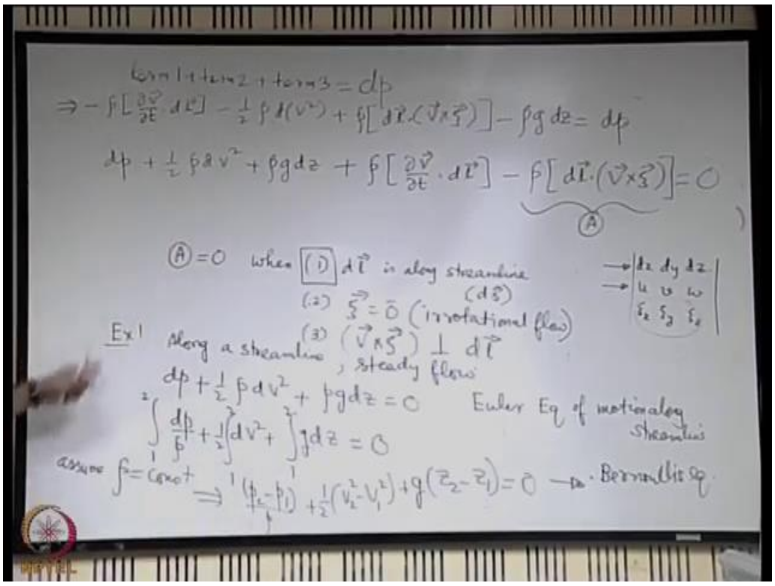
Term 3 $= -\rho g dz$

$$Term2 = -\rho\left[\frac{1}{2}\{\hat{i}\,\frac{\partial}{\partial x}\left(v^2\right) + \hat{j}\,\frac{\partial}{\partial y}\left(v^2\right) + \hat{k}\,\frac{\partial}{\partial z}\left(v^2\right)\}\right].\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) + \rho\left[(\vec{V}\times\vec{\zeta}).d\vec{l}\,\right]$$

$$= -\rho\left[\frac{1}{2}\{\frac{\partial}{\partial x}\left(v^2\right)dx + \frac{\partial}{\partial y}\left(v^2\right)dy + \frac{\partial}{\partial z}\left(v^2\right)dz\}\right] + \rho\left[(\vec{V}\times\vec{\zeta}).d\vec{l}\,\right]$$

$$= -\frac{1}{2}\rho d(v^2) + \rho\left[(\vec{V}\times\vec{\zeta}).d\vec{l}\,\right]$$

(Refer Slide Time: 16:34)



$$\Rightarrow -\rho\left[\frac{\partial \vec{v}}{\partial t}.d\vec{l}\right]-\frac{1}{2}\rho d\left(v^2\right)+\rho\left[d\vec{l}.\left(\vec{V}\times\vec{\zeta}\right)\right]-\rho g dz = dp$$

$$dp+\frac{1}{2}\rho d\left(v^2\right)+\rho g dz+\rho\left[\frac{\partial \vec{v}}{\partial t}.d\vec{l}\right]-\rho\left[d\vec{l}.\left(\vec{V}\times\vec{\zeta}\right)\right]=0$$

We will put the first important attention on the last term, because this particular term in a case when it is a steady flow, this trivially goes away. So, there is no big controversy or there is no big uncertainty in that that is quite understandable. But, the last term there are many possibilities, when the last term can become 0. What are the cases, so if you just write it in a determinant form, when you are having such a scalar triple product, you can write it in terms of determinants where each row of the determinant will represent the components of the vectors taken in the particular order.

$A = 0 \Rightarrow$ when (i) $d\vec{l}$ is along streamline

So, if they are located in the same direction that means, they are parallel vectors. So, if dl is located along a streamline, then we do not care whether it is a rotation or irrotational flow. But, if it is not, then if the vorticity vector is identically equal to 0, then A will become 0. No matter whatever is like no matter, whether dl is located along the streamline or not.

So, vorticity vector is a null vector, this is irrotational flow. And then sum of these three is 0 that means, if you integrate that, the integration will give a constant of integration. And that is what we actually saw in the example the problem that we discussed before going through this derivation.

$(ii)\vec{\zeta} = 0(irrotational$ flow)

$(iii)\left(\vec{V}\times\vec{\zeta}\right)$ is perpendicular to $d\vec{l}$

But, this mathematically you cannot rule out, you have a vorticity vector, you have velocity vector, you can find the cross product. And take an element in a direction which is oriented along that cross product. And then if you take such an element, then for such an element also for steady flow, it will appear that the Bernoulli type of equation is valid. So, this is not a Bernoulli type of equation.

This is in fact one step before that where we do not make any explicit assumption on how the rho or the density varies. So, this is still the Euler equation of motion. So, this is a more general way of writing the Euler equation of motion, where you are considering all the individual components, and trying to write that in a vector form, but at least we can understand that this term becomes 0 under which cases.

So, let us say that we are considering one such case. Let us say that we take an example, we are considering along a stream line. So, it relates change in pressure, change in velocity, change in elevation with respect to a change in position vector from point 1 to point 2.

So, when we are considering along a streamline that means, we are interested to evaluate that change by moving along a streamline.

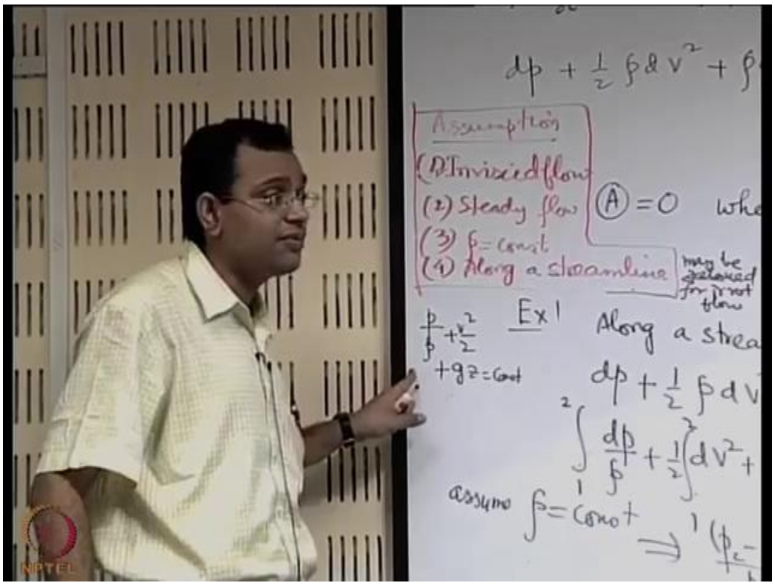
$$dp + \frac{1}{2}\rho dv^2 + \rho g dz = 0$$

$$\int \frac{dp}{\rho} + \frac{1}{2}\int dv^2 + \int g dz \ =0$$

$$\Rightarrow \frac{(P_2 - P_1)}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(Z_2 - Z_1) = 0$$

Now, you tell that, what are the assumptions that we followed in deriving this, so this is the Bernoulli's equation. We will come into the physical significance of this Bernoulli's equation in the next lecture, but let us at least try to identify that what are the assumptions that we utilize to derive these. So, first start with the most basic assumption

(Refer Slide Time: 28:36)



Assumptions

(1) Inviscid flow
(2) Steady flow
(3) ρ =const.
(4) Along a streamline

These are the four assumptions that we have considered in deriving this. Now, these are the assumptions that we commonly use, because commonly we utilize the Bernoulli's equation along a streamline. At the same time we must understand that these are not always the cases, inviscid flow is the most important thing.

Density equal to constant is not necessary. So, density equal to constant is the additional assumption beyond the Euler's equation. The most important assumption is in viscid flow. Because, many times we tend to apply the Bernoulli's equation in cases, when viscous effects are very much present.

The summary is if it is an irrotational flow, and other conditions are satisfied that is in viscid steady and ρ =const $\Rightarrow \frac{P}{\rho}+\frac{V^2}{2}+gZ=const.$

So, points 1 and 2 may be located anywhere in the flow field, still this equation is satisfied, if it is an irrotational flow. If it is not an irrotational flow, then 1 and 2 have to be located along the same streamline. So, these are very, very important fundamental assumptions that go behind the Bernoulli's equation.