Introduction to Fluid Mechanics

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Lecture – 31
Euler's equation

Today we are going to start with the discussion on Dynamics of inviscid flows.

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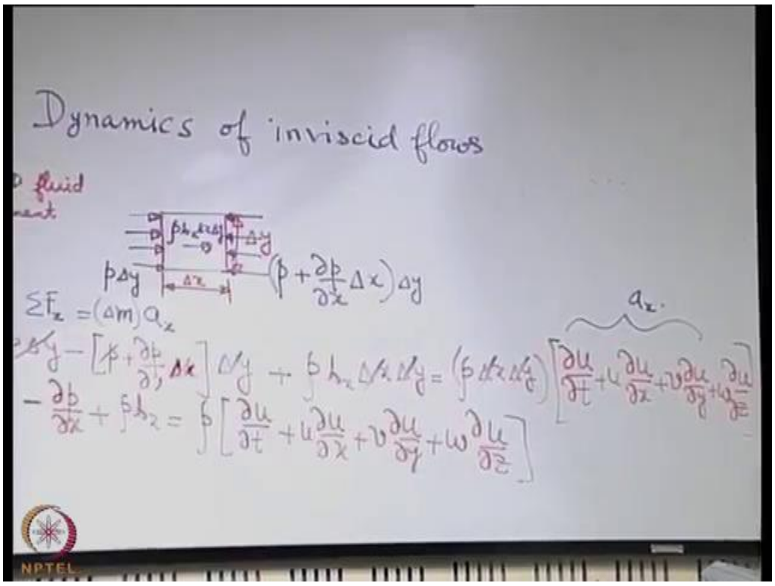

In the last chapter we were discussing about the kinematics. So, we were not discussing about the forcing parameters, which are involved to influence the flow. So, we have discussed about the motion. Now we are going to see that what are the forcing parameters which influence the motion and how they are related to the motion. When we talk about inviscid flows what we essentially mean is, initially we will discuss about cases where viscous forces are not present. So, it is a simplified situation of the reality. But at the same time it will provide us with a lot of important insight which we will use later on when we will be discussing about the Dynamics of viscous flows.

So, when we will be considering or focusing our attention in this particular chapter, we will be considering cases when viscous forces are not there, or negligible. To start with the discussion on this, what we will try to do, We will try to write the equation of motion for a fluid element where viscous forces are not present. So, when viscous forces are not present the kinds of forces which are there are the surface forces in terms of the normal components which are manifested through pressure. And some body forces which may be like the gravity forces. Keeping that in mind, let us say that we want to write, the equation of motion for a fluid element. Let us say that it is a two dimensional fluid element.

It need not always be two dimensional, but if we are writing the equation of motion along a particular direction then like for simplicity we can take it as a two dimensional one for illustration. So, let us say that we take a 2-D fluid element. As an example, fundamentally it is always three dimensional. So, the third dimension you may consider as one or some uniform third dimension. Let us say that these dimensions are Δ x and Δ y. We will quickly identify what are the forces which are acting on the fluid element only along x. So, we will identify forces along x, because we are interested to write the equation of motion along x.

So, other forces will not show. So, it is not a complete free body diagram only the x component of forces will be shown. So, here you have force due to pressure. So, if P is the pressure here. Then $P\Delta y.1$ where 1 is the width is the force that acts on the left phase, due to pressure. Force that acts on the right phase due to pressure, is $\left(P+\frac{\partial P}{\partial x}\Delta x\right)\Delta y$ . . Along x these are the only surface forces because other phases will have surface forces along y.

Body force may be there, let us say that b x is the body force per unit mass. $\rho b_x \Delta x \Delta y$ is the body force component along x. So, we can write the Newton's second law of motion for the fluid element. That is we can write resultant force along x, $\sum F_x = (\Delta m) a_x$.

$$P\Delta y - \left[P+\frac{\partial P}{\partial x}\Delta x\right]\Delta y + \rho b_x \Delta x \Delta y = \rho \Delta x \Delta y \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

See when we were discussing about the rigid body type of motion of fluid elements, then we did not use this expression.

We were using as an expression as if the entire fluid is having a particular acceleration. This is regarding the deformation within it. So, now, the different gradients of velocity will become important. Now we are more detailing it. So, we are looking into the detailed expression that that reflects that acceleration. So, $\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$ is acceleration along x.

$$-\frac{\partial P}{\partial x} + \rho b_x = \rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

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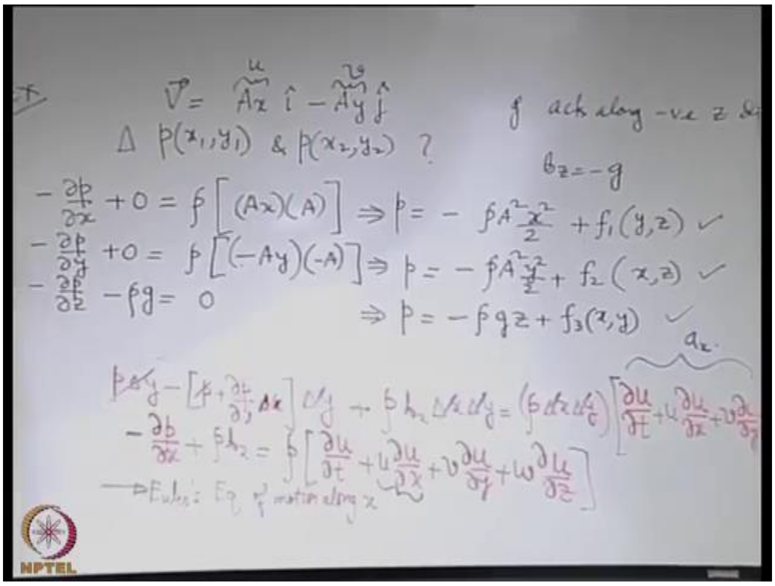

Similar expressions we can write for the motion along y and z. We are not repeating it because it is very trivial. Now what does this equation of motion contain? If you look into it, it is fundamentally like Newtons second law of motion. Where viscous forces are not considered. So, this right hand side is something like the mass into acceleration left hand side is the force effect of the force which is acting. So, one force is because of the pressure gradient and another force is because of the body force. These two forces are considered.

So, it is just a different way of like writing newtons second law of motion for a fluid where viscous effects are not present. And any other force other than this body force of this particular form we are not considering. Let us take an example to illustrate that how we can make use of this.

. Let us say that you have a velocity field V given by say $\vec{V} = Ax\hat{i} - Ay\hat{j}$. We are interested to find out, what is the difference between pressure at two points given by (x_1 , y_1) and (x_2 , y_2).

It is given that g, that is the acceleration due to gravity, acts along negative z direction. So, the question is what is the difference in pressure between these two points? The problem is very simple, but it will at least give us some idea of how to make use of this expression. A and B are not functions of time. So, it is a steady flow field. Let us write this equation say along x for this one. So, if we want to write this Euler's Equation along x.

$$-\frac{\partial P}{\partial x}+0=\rho\left[(Ax)(A)\right]$$

$$-\frac{\partial P}{\partial y}+0=\rho\left[(-Ay)(-A)\right]$$

$$-\frac{\partial P}{\partial x}-\rho g=0$$

So, it is possible to integrate these expressions, to find out how p varies with x, y and z.

Integrating $-\frac{\partial P}{\partial x}+0=\rho\left[(Ax)(A)\right]\Rightarrow P=-\rho A^2\frac{x^2}{2}+f_1(y,z)$

$$-\frac{\partial P}{\partial y}+0=\rho\left[(-Ay)(-A)\right]\Rightarrow P=-\rho A^2\frac{y^2}{2}+f_2(x,z)$$

$$-\frac{\partial P}{\partial x}-\rho g=0\Rightarrow P=-\rho gz+f_3(x,y)$$

All these three expressions are representing the same pressure field. So, we can compare these, to get these three functions. So, let us compare these and get the three functions.

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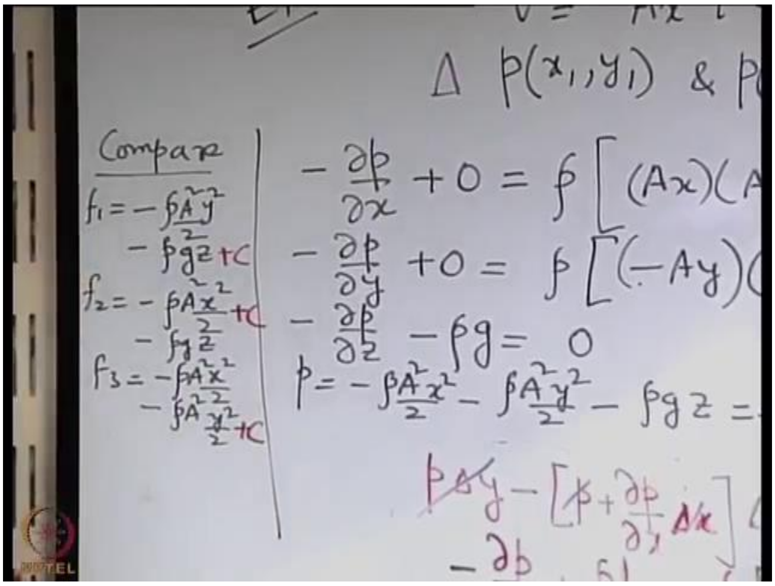

$$f_1 = -\frac{\rho A^2 y^2}{2} - \rho g z + C$$

$$f_2 = -\frac{\rho A^2 x^2}{2} - \rho g z + C$$

$$f_3 = -\frac{\rho A^2 x^2}{2} - \frac{\rho A^2 y^2}{2} + C$$

$$P = -\frac{\rho A^2 x^2}{2} - \frac{\rho A^2 y^2}{2} - \rho g z = -\frac{1}{2}\rho V^2 - \rho g z + C \Rightarrow P + \frac{1}{2}\rho V^2 + \rho g z = C$$

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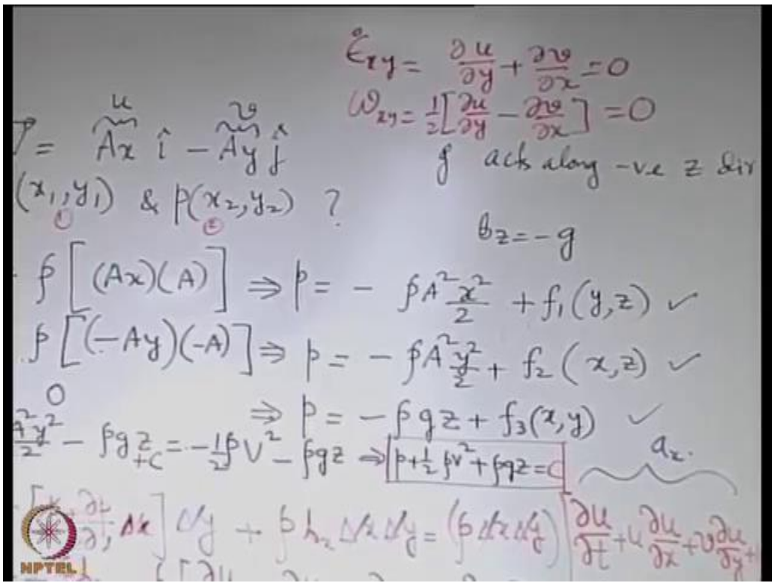

The direct comparison of the equation is possible because they represent the same pressure.

$$\dot{\varepsilon}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\omega_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] = 0$$

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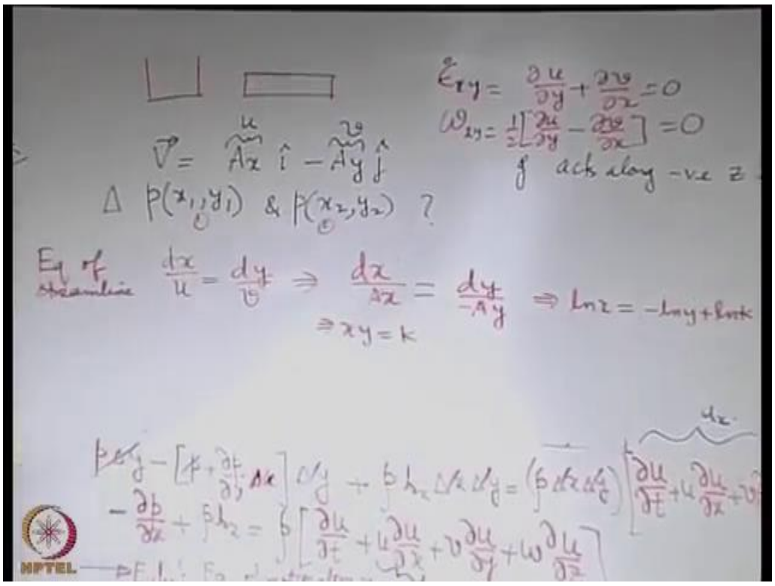

So, it is like if the fluid element was originally like this, maybe it will become ones like that and it will it will change its configuration in such a way that angularly there is no change. Only there are changes in linear dimension but ensuring incompressibility. Because it also represents an incompressible flow field that you can check by checking that divergence of the velocity vector is 0. So, it is an incompressible flow. So, that is one important observation. So, the important observation is it does not have any shear effect, it also does not have any angular velocity. So, it is like an irrotational flow because an irrotational flow has no angular velocity or no vorticity so to say.

$\frac{dx}{u} = \frac{dy}{v}$ is the equation of the streamline.

$$\frac{dx}{Ax} = \frac{dy}{-Ay} \Rightarrow \ln x = -\ln y + \ln k$$

So, this gives an equation of the streamline of the form xy=k which is like a rectangular hyperbola.

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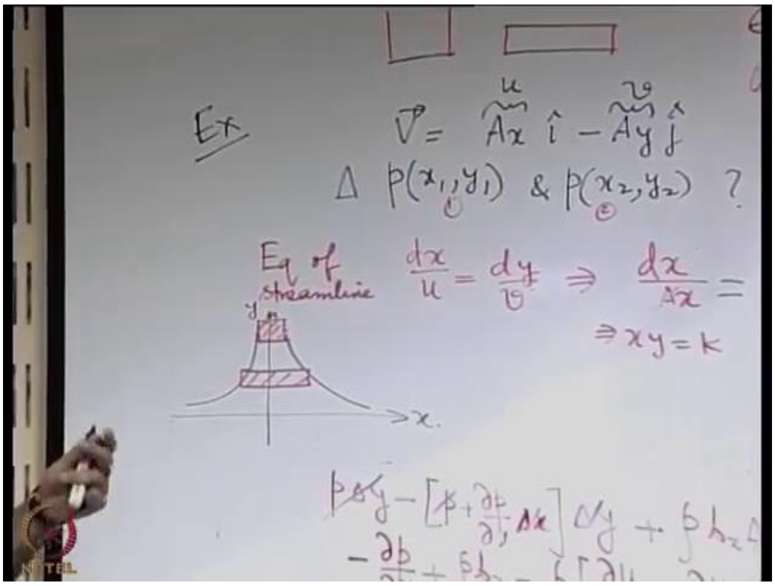

That means, if you have say, if you consider this as x axis and maybe this as y axis. It is possible that you have your streamlines in this way. So, if you have a fluid element, originally like this maybe the fluid element is coming down along the streamline. So, it is a case of pure linear deformation. No angular deformation ok. And then in such a case we have two things satisfied. One is there is no effective viscous effect because the viscous effect comes through

Viscosity $\times$ rate of shear deformation.

So, if the rate of shear deformation is 0, it does not matter whether viscosity is 0 or not. So, inviscid effect is not always through viscosity equal to 0. It may be the rate of shear deformation equal to 0 because eventually we are interested about whether the shear stress is 0 or not. So, if the shear stress is 0, it does not matter whether it is 0 because of mu equal to 0, or because of rate of shear deformation equal to 0. Here it is 0 because the rate of shear deformation is 0. So, it does not have any effect of this viscous shear, it does not have any effect of rotationality.

So, for it is effectively like an inviscid and irrotational flow and for such a flow, we will show later on that you can apply Bernoulli's Equation between any two points, in the flow field.