Introduction to Fluid Mechanics

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Lecture – 30
Circulation, Velocity potential

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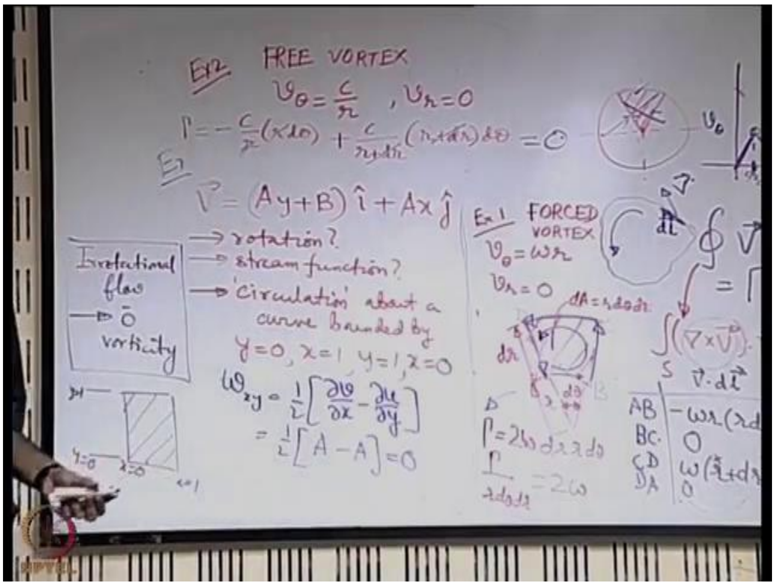

$$\vec{V} = (Ay + B)\hat{i} + Ax\hat{j}$$

You have to find out the fluid rotation and the stream function. Again, this is a very simple case to talk about, but whenever we are trying to understand the fluid rotation, we will now try to understand a very important concept which is asked in this particular problem. That is, what is the circulation about the curve? We will now learn what is the meaning of circulation about a curve bounded by the following lines y = 0, x=1, y=1, x=0?

So, we have come across at new terminology in this problem called as circulation. So, we will keep this problem a bit aside, try to learn what is the meaning of the terminology circulation and then we will try to apply it. The terminology circulation is not a very new terminology. It is very much related to what we have already learnt. So, and we will see that it is very much related to the concept of vorticity.

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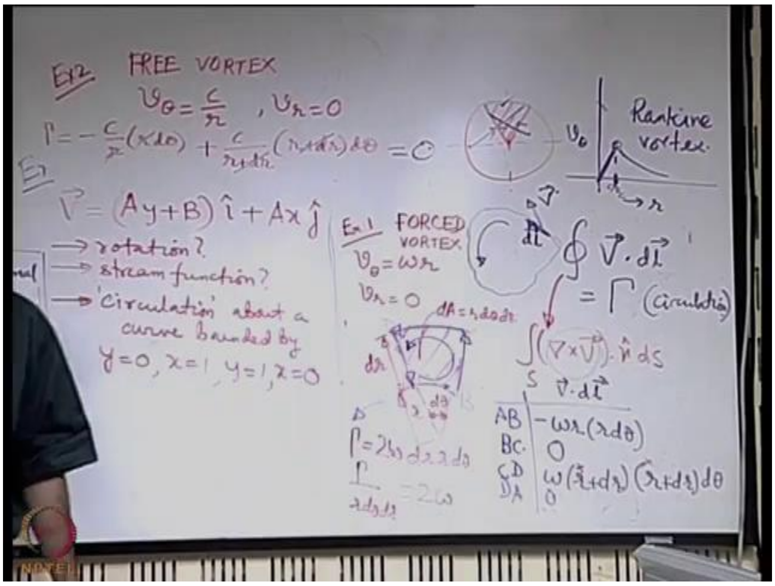

Let us say that we have a line a closed contour; a contour which is closed by some curve.

Now, if you take a small line element located on the contour, let us say that we take a line element like this. So, you may have a velocity vector; say v and you may assign a vector to this line element by giving it a directionality. Say you are traversing along a clockwise direction or an anti-clockwise direction. Here, we are traversing along an anti clockwise direction.

$$\oint \vec{V}.d\vec{l} = \Gamma$$

$\int_S (\nabla \times \vec{V}).\hat{n}dS$. So, you can express this in terms of curve X or here the curve V where the vector function f is here V where s is an element of the surface d s is an element of the surface and s is that total surface that is bounded by this closed curve that is very important.

So, you can clearly see that if you talk about the value in a plane, it is possible to make out from this that first observation is this is nothing but the vorticity vector. So, this Vorticity vector when you utilize this vorticity vector here, so the circulation is nothing but how you express it in terms of vorticity, it is just like roughly vorticity per unit area. Or it is in the other way, it is like vorticity into the area is the circulation. So, it is just the other way not vorticity per unit area, but vorticity $\times$ area is the circulation.

So, let us take an example before going into this particular mathematical form to figure out that how you can calculate what is the circulation. Now, to do that, let us take an example which is not the same example but we will utilize what we learn from that example to solve this problem. Say you have another example where you have the velocity components given by polar coordinates, $\begin{matrix} v_\theta = wr \\ v_r = 0 \end{matrix}$

Let us say that we want to find out what is the circulation around the closed contour. So, a closed contour in a element in a polar coordinate form.

So, this like A, B, C, D, now we are interested to find out the circulation. So, we are first we start the traversal from the point A. So, we are going from A to B in this direction.

So, when you go from A to B, vθ is opposite to dl that is drawn. So, vθ.dl will give a - sign. So, for AB $-wr(rd\theta)$.

Then for BC it is 0.

For CD it is $w\left(r+dr\right)(r+dr)d\theta$

For DA it will be 0.

$$\Gamma = 2\omega r dr d\theta$$

$$\frac{\Gamma}{r d\theta dr} = 2\omega$$

So, this is like the vorticity. So, what we get from here this is an example of illustration of the same concept, this is just from a vector calculus theorem, this is just detailed illustration of that. So, you can conclude from this that in such a case, you can find out the circulation per unit area equal to vorticity. So, from that you can relate vorticity with circulation. So, if you find out one, you can automatically find out the other.

This type of example where you have $v_\theta =$ r and $v_r = 0$, this is known as a Forced Vortex. So, it is something like you are creating a rigid body type of rotation in a flow by having an angular velocity to the system. We have discussed in the class earlier that you have a cylindrical tank and you start rotating the cylindrical tank in the limit as you are neglecting the viscosity, it takes the shape of a paraboloid of revolution type, that is an example of a forced vortex.

So, the circulation per unit area is given by the vorticity. Let us say that somebody has taken an area like this. So, this is a circle, somebody has taken an area like this. This is the center of the circle and is trying to find out what is the circulation. We will see that in certain cases, this definition will be restricted if we include the origin. So, let us take another example that will illustrate that what is the corresponding problem. So, take an example 2. With understanding of example 1 and example 2, this example problem that we are going to solve it will be easy for us to appreciate.

Example 2 is something which we call as free vortex. What is the free vortex? So, free vortex is defined as $v_\theta = \frac{c}{r}$. It is directly proportional to the radius it is inversely proportional to the radius and $v_r=0$. So, this kind of a situation you get in a kitchen sink. So, you open the tap and what you will see, that as it comes, so the sink is first say you fill up, the sink first with water close the valve so that the water cannot go out.

Now, suddenly open the valve you will see that as the water comes to the outlet and goes through the pipe. When it comes to the center of the sink, it is coming out with a very high velocity. So, there and it has the sense of rotationality in the flow. So, v_θ is inversely proportional to the radius. It is singular at r = 0 mathematically because at r = 0, it is as if $v_\theta \to \infty$ and you cannot have an infinite velocity.

$$\Gamma = \frac{C}{r}(rd\theta) + \frac{C}{r+dr}(r+dr)d\theta = 0$$

These types of cases where the vorticity is 0 is known as irrotational flow. So, irrotational flow is a flow with 0 vorticity. We will come into the concept of irrotational flow soon. Irrotational flow means 0 or null vorticity vector that is the very simple definition. And from the name itself, it is quite clear that there is no element of rotationality in the flow; that is why it is called as irrotational flow.

Now, if in this example, you take a closed contour like this this. By definition is a closed contour, you try to find out what is the circulation. So, for these two edges, the circulation will not be there. For this edge, the circulation will be there and clearly, when you sum it up over the contour, it is not 0 if you take such an element. What is wrong with that a very important conceptual mistake is, there by choosing this element you have taken the element by including the point of singularity. So, you cannot take an element which contains the point of singularity to establish the relationship between the circulation and the vorticity.

So, the practical case if you draw the velocity profile v_θ vs r if you use a completely free vortex understanding, then v_θ will be undefined. So, actually always very close to r = 0 it is a force vortex. So, close to r = 0, $v_\theta = wr$. So, it will be something like this and away from that, it may become a free vortex.

So, it is like a rectangular hyperbola type.

There is a rotation close to this one. So, this is known as the, this is the practical example. Free vortex or forced vortex, neither of these are like very practical examples, but their combinations are quite practical. This is known as Rankine vortex. So, it is a combination of free and forced vortex. So, up to a particular radius say critical radius, it is like a forced vortex. Beyond that, it is like a free vortex a very classical example that occurs in nature is a tornado.

So, a tornado very close to the eye of the tornado, it is up to that it is a force vortex. So, it has a strong element of rotationality; beyond that, it is like a free vortex. So, a tornado may be very well approximated by a Rankine vertex which is a combination of free and forced vortex. Now, keep in mind that if you have such a combination, you have to satisfy that at this point at these critical radius where you have like a transition of behavior from free to force vortex the v_θ should be same as given by the considerations of free and force vortex because you cannot have a discontinuous velocity field. The velocity field in the physical sense is continuous.

Now, if we come to this example, we will try to utilize this example to find out the circulation. So, the rotation is given by the first you find out the curl of the velocity vector. So, it is a 2 dimensional field.

$$\omega_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{1}{2}\left[A - A\right] = 0$$

The rotation is 0; that means, it is vorticity is 0 and by the relationship between circulation and vorticity, now you have a curve bounded by what y = 0 and 1 and x = 0 and 1.

So, it is like a rectangular contour. This is x = 0, this is x = 1, this is y = 0, this is y = 1. So, this is the contour. This is the area bound bounding the contour. So, this does not include any point of singularity. So, this is a valid area and therefore, the circulation about this curve is expressed in terms of the vorticity.

Next, we will go to concept that is very much related to the rotationality the flow again and that concept is given by the name of velocity potential. We will see that what is the velocity potential and what are the important considerations that go behind this.

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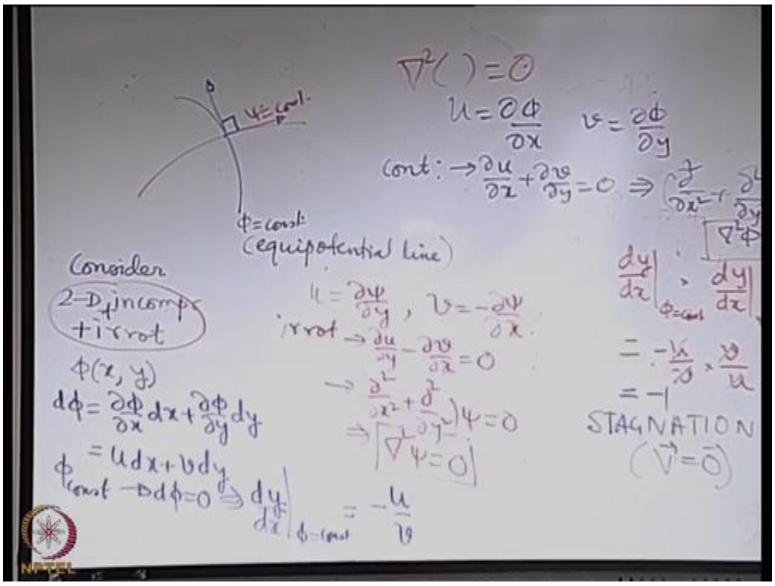

Let us consider the case of irrotational flow. So, this is defined only for irrotational flow. So, for nothing else, therefore, we must find out the condition for irrotationality. So, what is the condition for irrotationality? The curl of the velocity vector is a null vector $\nabla \times \vec{v} = 0$.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$
$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0$$

So, from this, you can see that if you now parameterize say

$$u = \frac{\partial \phi}{\partial x}$$
$$v = \frac{\partial \phi}{\partial y}$$
$$w = \frac{\partial \phi}{\partial z}$$

It is in general a function of x, y, z, then this definition automatically satisfies this requirement. This is very much obvious because from the theory of vector fields, you know that if the curl of a vector is a null vector, then that vector may be represented as the gradient of a scalar function.

So, if the curl is 0, it is a null vector. $\vec{v} = \nabla \phi$. Now, such a field where the curl of the field is null vector, the field what we talk about is a general vector field, here it is a velocity vector field we call it as a conservative velocity field.

So, in general in field theory if the curl of a particular vector field is a null vector, that field is a conservative field. It may be expressed in the form of gradient of a scalar potential.

The work done for going from 1 to 2 is independent of the path and just is dependent on the difference in the potential that is the potential energy in this case. So, that path dependence comes from the conservative nature of the field.

So, in a conservative field, so similarly this is talking about a conservative velocity field, not a conservative force field, but the concept is very much analogous. So, whenever you have such a conservative field, the field is expressible in form of gradient of a potential. In such, particle mechanics in a gravitational field that potential is the potential energy that we know. So, here we are talking about a velocity potential.

So, now, we are interested say about a relationship between the velocity potential and the stream function. Now, when we are interested about a relationship, they must be comparable. So, when under what case, under what circumstances they are comparable. Now, objective is to find out a relationship between stream function and the velocity potentials of what type of flow should we considered consider what.

So, we should consider the case when both are defined. So, when is stream function defined? It is defined for 2 dimensional incompressible flow. When the velocity potential is defined for irrotational flow? So, it has to be 2 D + incompressible + irrotational so that, both $\Phi \& \psi$ functions are defined.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy$$

$$\phi_{const} \to d\phi = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{\phi=const} = -\frac{u}{v}$$

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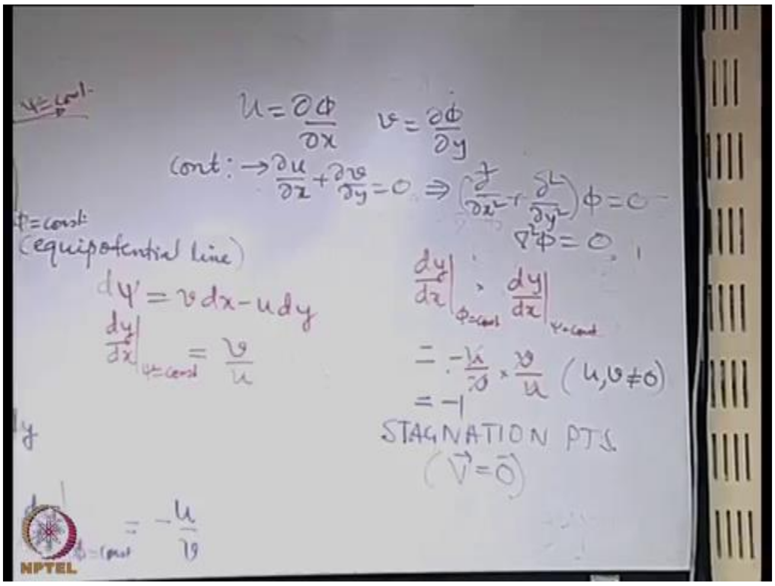

$$d\psi = vdx - udy$$

$$\left.\frac{dy}{dx}\right|_{v=const} = \frac{v}{u}$$

Now, if at a common (x , y), $\left.\frac{dy}{dx}\right|_{\phi=const} \times \left.\frac{dy}{dx}\right|_{\psi=const} = -\frac{u}{v} \times \frac{v}{u} = -1$

If at least one of these components, you have 0, then you get division by 0. So, you have to make sure that these are nonzero. So, if these are non-zero, then only you may cancel out this and when you are cancelling out both you are ensuring that u v both are nonzero because both appear in the denominator.

So, here you have a tangent like this and here you have a tangent like this. So, these are perpendicular to each other. And therefore, the important conclusion is that ϕ equal to constant lines, which are called as equipotential lines. So, this is equipotential line. So, this is very much analogous to the equipotential line that you get in say electromagnetic field theory.

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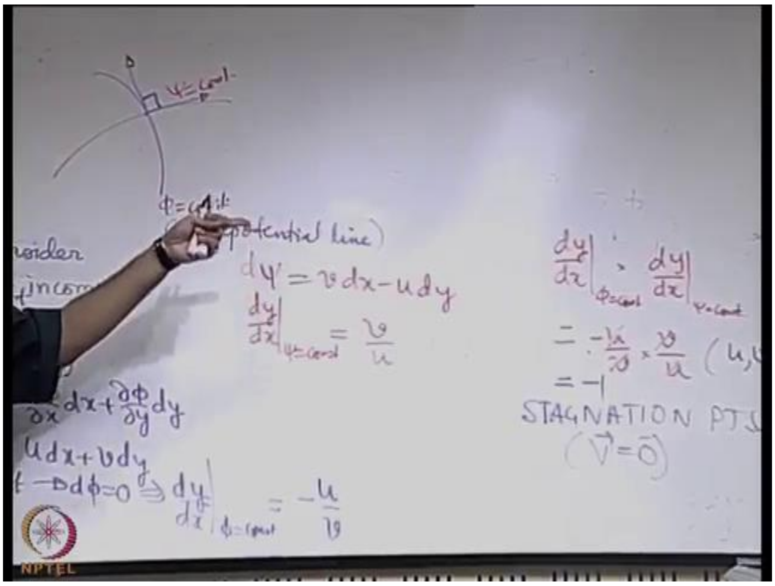

So, here you have a tangent like this and here you have a tangent like this. So, these are perpendicular to each other. And therefore, the important conclusion is that $\phi = const$ lines, which are called as equipotential lines. So, this is equipotential line. So, this is very much analogous to the equipotential line that you get in say electromagnetic field theory.

So, the $\phi = const$ and $\psi = const$, these lines are orthogonal to each other everywhere in the flow field except for certain points. The points are where the velocities are 0; those points are known as stagnation points. So, in a flow field the points where the velocities are 0, those are called as stagnation points. So, stagnation point is a point where v is 0 and at the stagnation point, you do not have such a relationship because at a stagnation point you cannot really work out these. So, it is not true that the streamlines and equipotential lines are orthogonal everywhere in the flow field. They are orthogonal at each and every point except for the stagnation point where it is not defined in that way.

Now, finally, what we will see, we will see a relationship or a governing equation for the expressions for ϕ and ψ . So, if you say that you are interested about this case when you have both 2 dimensional incompressible and irrotational flow. So, when you have a 2 dimensional incompressible flow, so you have say u irrotational flow you have $u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$.

$$Cont :\rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \Rightarrow \nabla^2 \phi = 0$$

Similarly, if you start with the definition of the stream function, $u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$

$$Irrotational \rightarrow \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = 0$$

$$\nabla^2 \psi = 0$$

So, both ϕ and ψ satisfy the Laplace equation. It does not mean that their solutions are same because boundary conditions are different. All the governing equations for ϕ and ψ are the same, but their boundary conditions are different and therefore, solutions are different. So, in summary what we can see, that we have defined what is the stream function, we have defined what is the velocity potential, we have seen that there is a relationship between these two when we have both defined that is 2 dimensional incompressible irrotational flow and both in terms of their governing equations and also in terms of their orthogonality.

Now, what we will do finally, we will look into some visual demonstration of certain types of flows and end up the discussion today. So, in these visual demonstrations, we will see some example.

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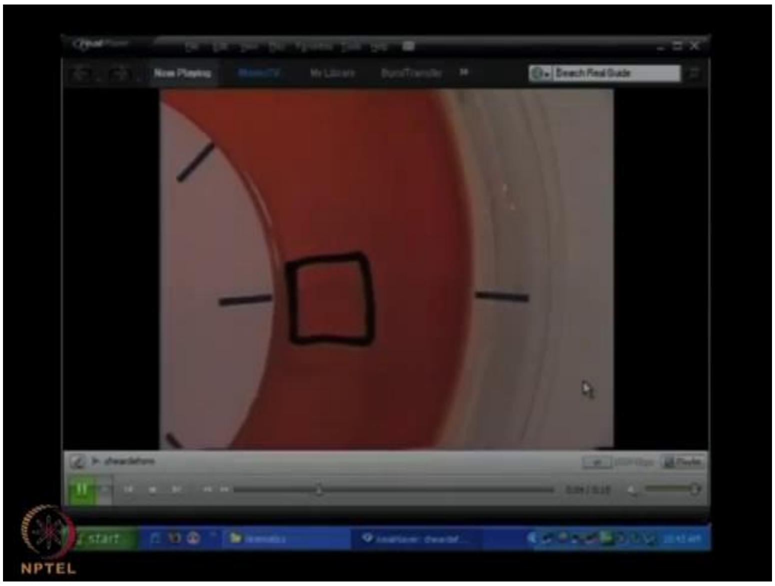

So, first you see, this is a this is a shear deformation see if there is a line element which is marked and this is a flow between two concentric cylinders. We have seen such a case when we are discussing about viscosity. So, you can clearly see that the fluid element is deforming. So, from a rectangular shape, the fluid element is coming to the deformed shape. So, this is the angular deformation that we are talking about, we just play it again, so that you can see it again. So, originally it was rectangular, but because of the shear you see that how it is getting deformed.

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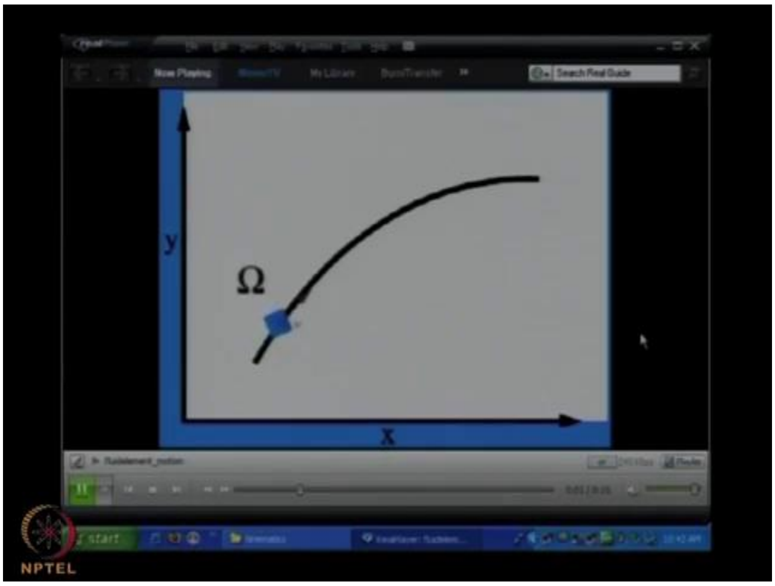

Now, we will look into a second example after this where we will see that like if you have a general case, when the when a fluid element is moving along a path then like it can have a general type of behavior. So, it can have deformation. So, here it is just rotating, you see that it is not deforming. So, it is just a pure rotation.

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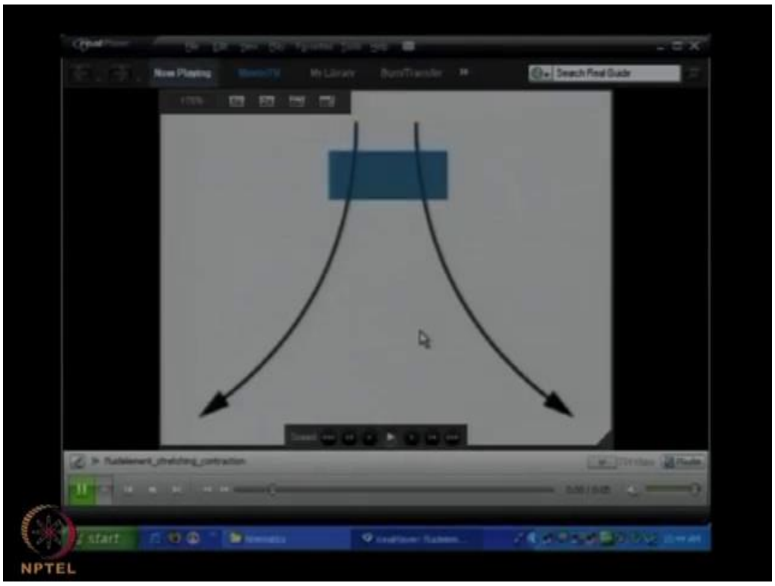

Now, let us look into a third example where we see a flow, this is a very interesting case. This is a case when it does not have any type of angular motion angular deformation neither shear deformation nor rotation. So, it is as if just it is getting stretched. So, I will leave it you to you on it as an exercise you find out what are the velocity components u and v, that will lead to this condition. What are the important restrictions? You must have both the partial derivative of u with respect to y and partial derivative of v with respect to x equal to 0, then both the angular velocity as well as the angular deformation will be equal to 0 and then such a case is visible, that is known as a stagnation point flow.

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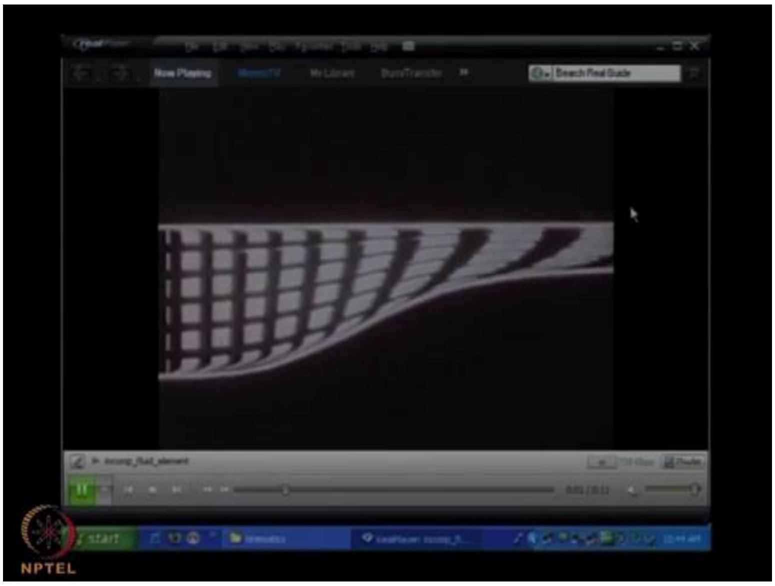

Finally, let us look into a general case of an incompressible fluid element, how these fluid elements flow. So, this is a visual demonstration of the continuity equation. So, you see that whatever flow is entering the same flow is leaving. So, if you have a constraint passage, what is happening is that the velocities are increasing to compensate for the decrease in the cross section area. So, this is what like $A_1u_1 = A_2u_2$ that type of expression that we have seen. So, we stop here today.