Introduction to Fluid Mechanics
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Lecture - 29
Stream function

We continue with our discussion on Stream function.

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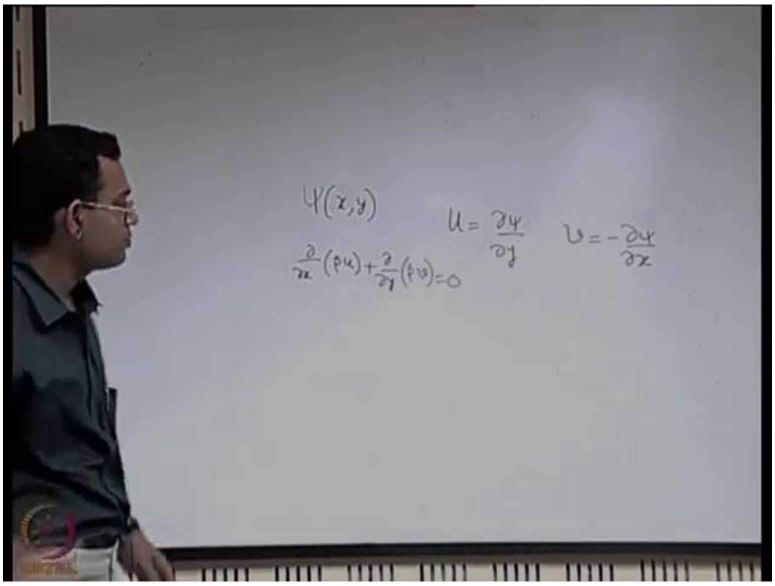

So, the stream function we defined as a function of x and y with an understanding that it should satisfy the requirement of mass conservation constrained by the continuity equation and special case is giving that parametric form 2 dimensional incompressible flow.

$$\psi(x, y)$$
$$u = \frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}$$

For other types of flow special cases again may be defined with a slightly adjusted manner. Now say we are interested to define it for a 2 dimensional compressible flow, but or a 2 dimensional steady flow may be compressible may not be compressible.

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

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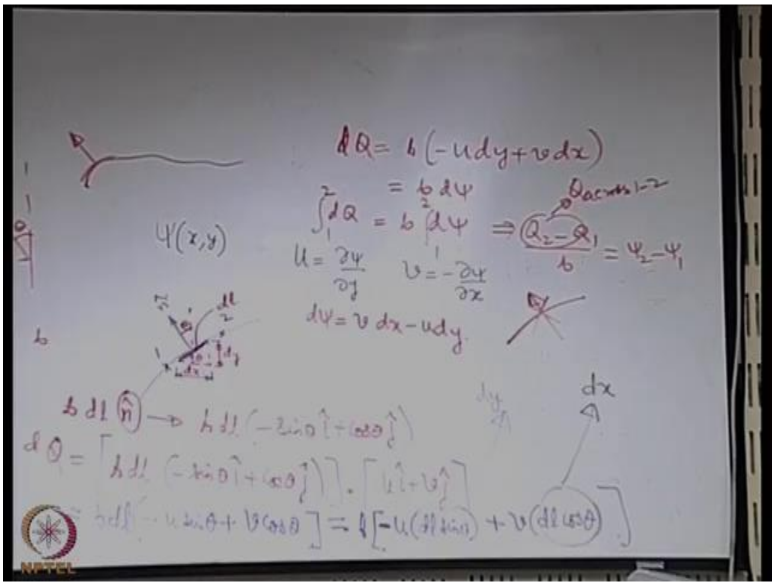

To understand understand that let us say that we have some line element and we are marking two points 1 and 2. We are interested to find out what is the total rate of flow between these points 1 and 2, how we can do that? To do that let us say that points 1 and 2 are located so closely that they are or maybe let us consider a small element within that domain bounded by points 1 and 2, that small element is given by a length of dl which is almost like a chord of this curve.

Now, let us say that the length of this is dl. So, if you consider it like a vector sense it has a outward normal vector like this, say $\hat{n}$ and it may be expressed as a function of the important angles. The length of this is dl let us say that b is the width perpendicular to the plane of the figure.

So, it is possible to write this dl and subsequently da which is dl.b and the orientation given by the angle θ . So, dl in a vector form is like dl.$\hat{n}$; it is just like the area in a vector form is the magnitude of the area, times the unit vector normal to the area and away out outward to the area. So, here the area is represented by dl, b is the width.

$$\hat{n} \rightarrow bdl(-\sin\theta\hat{i} + \cos\theta\hat{j})$$

Volume flow rate is the dot product of the velocity with the area because if you have an arbitrary area what will give rise to a volume fluid only that component of velocity which is perpendicular to the area that gives a net flow rate right.

So, if this is an area and the velocity vector may be arbitrary oriented, but its component normal to that direction of the area is what is only important; that means, dot product will give the component along that direction. So, the dot product of the velocity with the area vector will give the component of velocity along the area vector. Area vector means area normal; so that will the product of that will give the volume flow rate.

$$dQ = [b.dl(-\sin\theta\hat{i} + \cos\theta\hat{j})].[u\hat{i} + v\hat{j}]$$

We are assuming it is a 2 dimensional flow because we are discussing it in the context of stream function. So, everything is a 2 dimensional concept that we are discussing

$$dQ = [b.dl(-u\sin\theta + v\cos\theta] = b[-u(dl\sin\theta) + v(dl\cos\theta)]$$

$$dQ = b(-udy + vdx) = bd\psi$$

$$\int_1^2 dQ = b\int_1^2 d\psi \Rightarrow \frac{Q_2 - Q_1}{b} = \psi_2 - \psi_1$$

It is quite logical because we have defined stream lines in such a way that stream lines are such that the velocity vectors are tangential to the stream line.

So, a very important corollary of that is there cannot be any flow across a stream line. So, if you have a flow if you have a stream line like this you cannot have an any flow across this because all the velocity vectors are tangential to it; there is no normal component of velocity and normal component of velocity can only give a cross flow.

Streamlines are characterized by constant stream function. So, you do not have a change in stream function along a stream line and therefore, no flow across a stream line. So, the difference in stream function between 2 points gives a quantitative indication of what is the flow rate across a line element that joins those 2 points per unit width.

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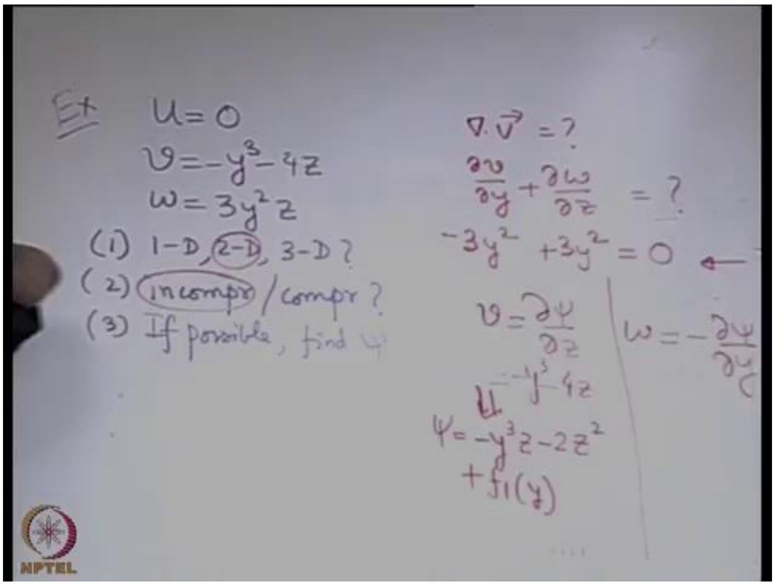

So, let us work out maybe one simple problem to begin with say you have flow field given by the following components

$$u = 0$$
$$v = y^3 - 4z$$
$$w = 3y^2 z$$

Now the questions are as follows, number 1 is the flow 1D, 2D or 3D? The next one whether it is incompressible or compressible flow and the third part is that if possible define a stream function for the flow.

So, the first part, it is a 2 dimensional flow because it has 2 velocity components.

The divergence of the velocity vector you should find out because that gives the rate of volumetric strain. So, if the rate of volumetric strain is 0; that means, it is incompressible flow. So, we have to check what is the divergence of the velocity vector?

$$\nabla \vec{v} = ?$$
$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = ?$$

$$-3y^2 + 3y^2 = 0 \leftarrow incompressible$$

So, now, you have a 2 D flow from the first part you have an incompressible flow from the second part; since it is 2 D and incompressible the stream function itself is defined. So, we can now attempt to find out the stream function.

$$v = \frac{\partial \psi}{\partial z}, w = -\frac{\partial \psi}{\partial y}$$

Now see that with a slight change the components of velocity given as y and z components instead of the previous ones you see that you are facing a dilemma that should not be there. Again the objective will be that given a form of the continuity equation; just write a parametric form which satisfies z and that will automatically give you the definition of the stream function for that case.

$$v = \frac{\partial \psi}{\partial z} = -y^3 - 4z$$

$$\psi = -y^3 z - 2z^2 + f_1(y)$$

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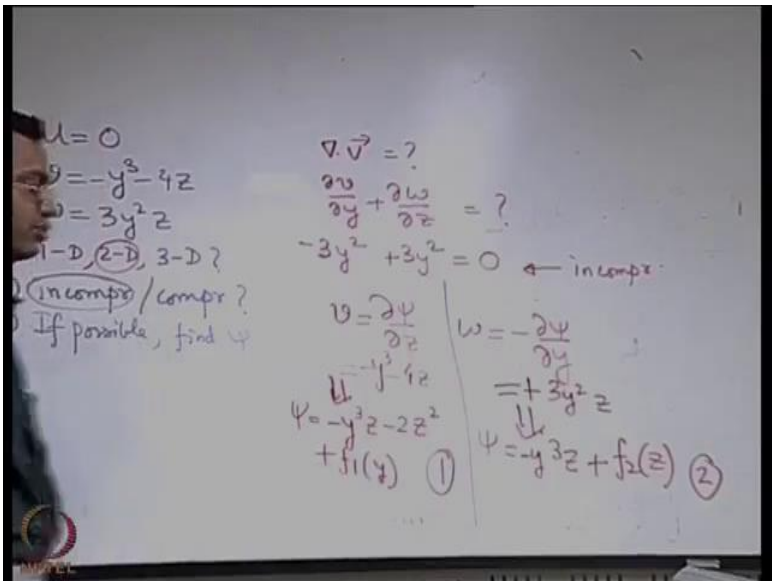

$$w = -\frac{\partial \psi}{\partial y} - -3y^2 z$$
$$\psi = -y^3 z + f_2(z)$$

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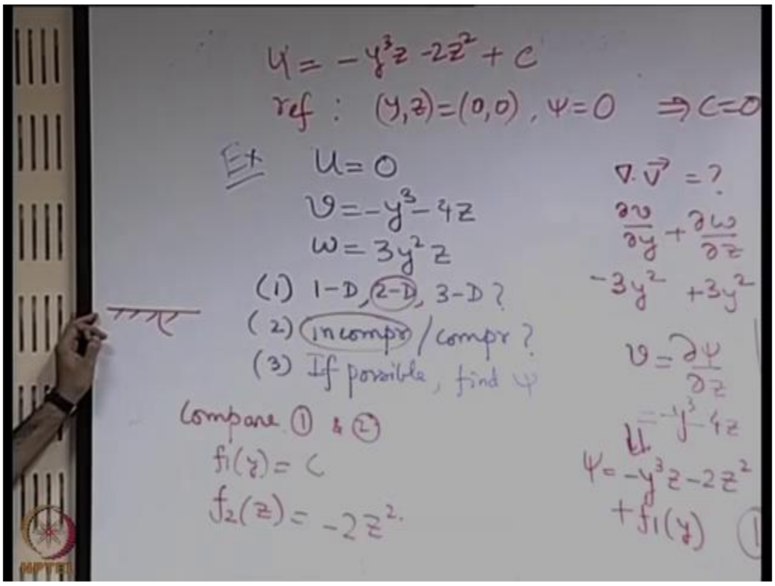

$f_1(y) = C$

$f_2(z) = -2z^2$

$\psi - y^3 z - 2z^2 + C$

(y,z)=(0,0), $\psi = 0$

But for working convenience you may set your references in that way in most practical purposes you have flows on solid boundaries. So, solid boundary itself is a stream line right because there is no flow across it. So, by its physical sense any shape solid boundary is itself a stream line. So, you can give it a value of a particular stream function; if it is a 2 dimensional incompressible flow.

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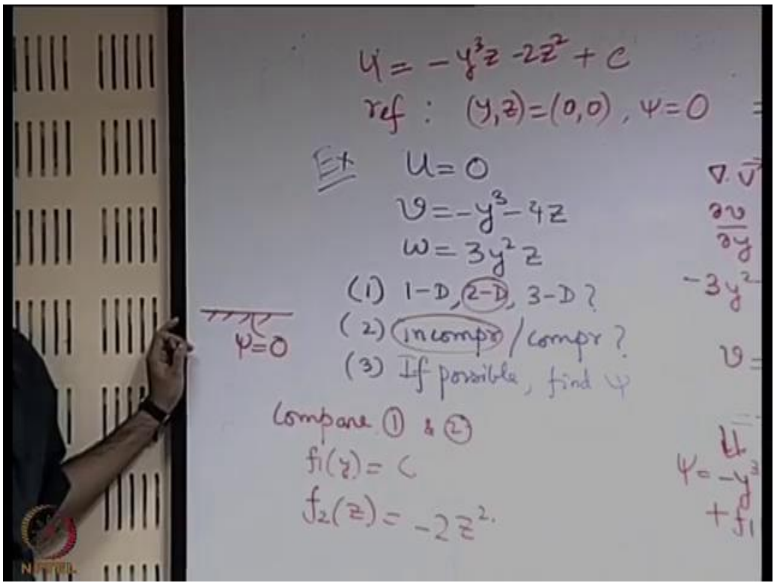

So classically just as a matter of convention we give each ψ equal to 0 as a reference. So, solid boundaries are classically referenced as 0 stream functions.

So, let us maybe move on to next problem which may be similar to this, but let us just work out another problem with respect to the concept of the stream function.