Introduction to Fluid Mechanics
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Lecture – 28
Deformation of fluid elements-Part-III

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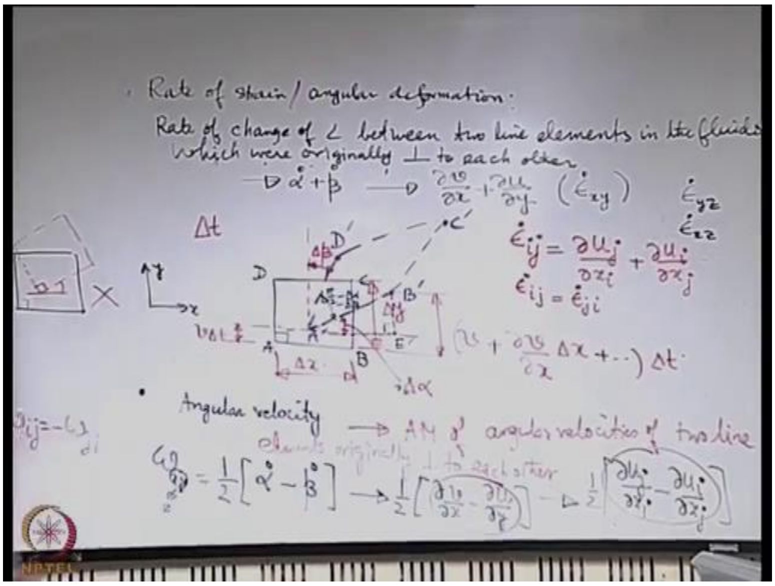

Now when you want to define the angular velocity of the fluid element, it is not such a straightforward picture as that of rate of strain or rate of angular deformation.

See whenever you are thinking of angular velocity we were discussing about the rigid body motion, but that is a special case. Usually fluid is not under a rigid body motion just like this case. So, when it is not in a rigid body motion you cannot really have a unique angular velocity of all the line elements in the flow.

So, the line elements say AB it has some angular velocity, the line element AD; it has a different angular velocity. That is why these angles alpha dot this angles delta alpha and delta beta they are different. If it was rotating like a rigid body then what would have been the case? Then the case would have been like this. So, if you have the original fluid element now you have the fluid element; may be like this. So, that this angle is preserved that is a rigid body rotation, but that is not happening here in at least in the example that we have drawn in the figure.

So, if that was the case we could clearly say that what is the rate of change of this angle and the time rate of change of that angle would have given the angular velocity, here those angles are different. And therefore, we have to come to our angular velocity definition with a compromise; not exactly same as we do for rigid body mechanics, but keeping in mind that in the special case that it becomes a rigid body, it should follow the rigid body mechanics definition of angular velocity.

So, when we say that we may define the angular velocity in this way that it may be thought of as the arithmetic average or arithmetic mean of the angular velocities of two line elements which were originally perpendicular to each other.

So, if you have a case where you have the fluid element having an angular deformation so that it is not a rigid body rotation. So, arithmetic mean of the angular velocities of the two line element; that means, half of the angular velocities of AB and AD which were originally perpendicular to each other. The angular velocity of AB is $\dot{\alpha}$, what is the angular velocity of AD?

According to this figure, it is $-\dot{\beta}$ because $\dot{\alpha}$ in this figure is in the anti clockwise direction, if we take that as positive $\dot{\beta}$ is in the clockwise direction we should take it as negative..

$$\omega_z = \frac{1}{2}\left[\dot{\alpha} - \dot{\beta}\right] \rightarrow \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Consider the special case when $\dot{\alpha} = -\dot{\beta}$; which is represented in this figure. Because when $\dot{\alpha}$ $= -\dot{\beta}$ that is even in terms of the total deformation, $\Delta\alpha = -\Delta\beta$ then $\Delta\alpha + \Delta\beta$ becomes 0. So, the angle which was originally $\frac{\pi}{2}$ remains same as $\frac{\pi}{2}$ and that is the rigid body deformation case that we are considering.

So, in that case when $\dot{\alpha} = -\dot{\beta}$ it becomes as good as either $\dot{\alpha}$ or $\dot{\beta}$. So, in that limit it corresponds to the definition of angular velocity in rigid body mechanics. So, whenever we are defining something; we have to keep in mind that in a special limit which is already known it should be consistent with that special limit; the definition should not violate that special case.

$$\omega_z = \frac{1}{2}\left[\dot{\alpha} - \dot{\beta}\right] \rightarrow \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{1}{2}\left[\frac{\partial u_j}{\partial x} - \frac{\partial u_i}{\partial y}\right]$$

$\frac{1}{2}$ factor is put as a matter of definition to ensure that in the limit when it is like a rotating like a rigid body the; its angular velocity is same as the definition of the angular velocity of a rigid body that is why that adjustment factor comes.

Now, if we see clearly we have come up with two different types of angular motion representations. One is through this $\dot{\varepsilon}_{ij}$, another is the angular velocity and it is possible to see that when they are combined with each other what do they actually represent.

If you write it in a matrix form it is a skew symmetric matrix. Now it is possible to combine these two deformations and write it in some way which is again a very straightforward thing.

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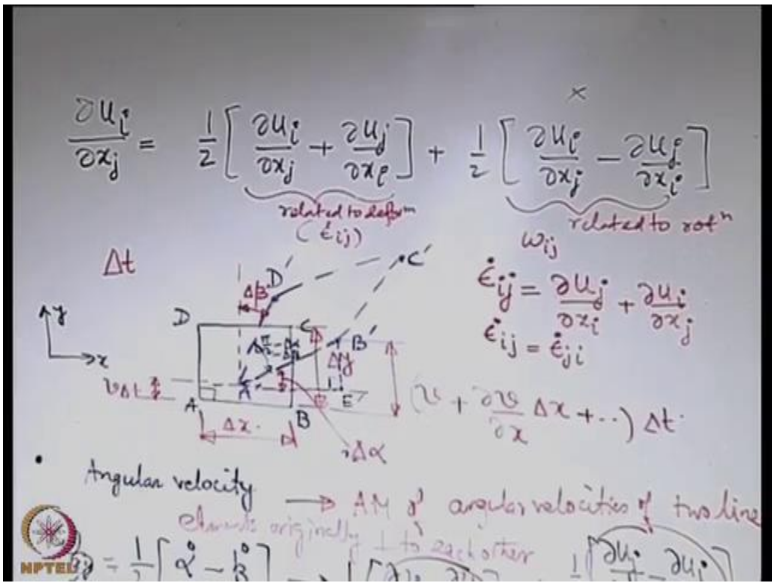

That is $\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$. This is a general velocity gradient and a general velocity gradient is what is related to what is expected to be related to deformation.

Now, you see that when you have a general velocity gradient out of that only one part is related to deformation. And another part is not related to deformation it is related to just angular motion like a rigid body. So $\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]$ is related to deformation and $\left[\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right]$ is related to rotation.

So, what we can see from here is a very important thing. So, when we write the general velocity gradient which should be in general a function or a parameterization of the deformation out of that clearly we distinguish that one part is not related to deformation. So, only $\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]$ is related to deformation and whenever you relate shear stress with the velocity gradient; $\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]$is what is important because this part is giving rise to angular deformation, the other part is a rigid body motion.

So, it should not be relating the constitution of shear stress in a material in a fluid. So, whenever we have discussed about the Newton's law of viscosity; you clearly see that this is the term that we had actually taken.

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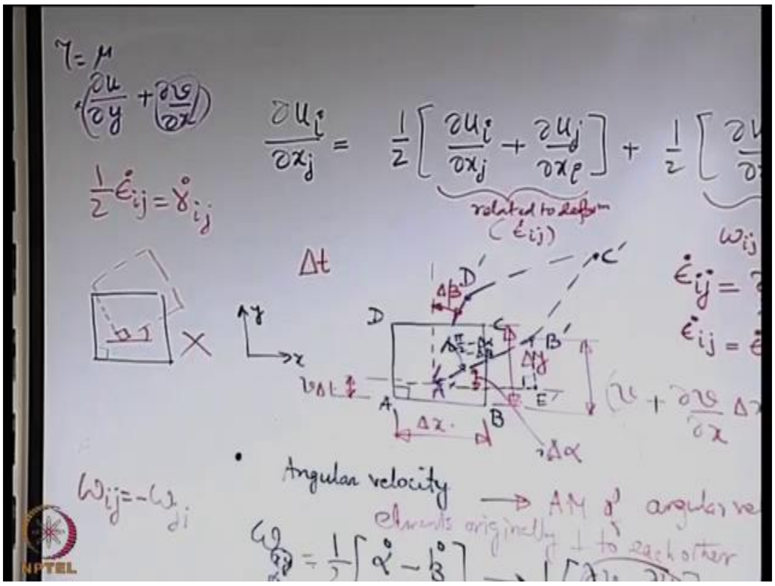

$\tau=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$. We were considering a unidirectional flow so v=0. So, whatever form of Newton's law of viscosity we have discussed in the fundamental way, it is correct that is the shear stress is linearly proportional to the rate of angular deformation, but the quantification of angular deformation that we made earlier was based on a very simple case and assumption.

So, as we advance and proceed more and more we will come into more and more decorous ways of writing the Newton’s law of viscosity. And we will again take it up later on in one of our chapters, we will not take it up in this chapter because these just bothers about the kinematics. So, this does not bother about the force.

In that special example which we took up in our earlier lectures we consider there is no other velocity components; therefore, those other terms were not appearing. The other important observation is that although; it looks something which is non trivial, but actually there cannot be a more trivial expression than this. Because it is just like you are writing a form every matrix can be written as a sum of a symmetric and a skew symmetric matrix and that is what we are trying to do.

So, eventually that symmetric matrix is being represented by the components of the angular deformation and the skew symmetric matrix is represented by the components of the rotation. So, in a more formal way since we are dealing with tensors. So, matrices are some of the ways by which we may represent say a second order tensor in a notational form. So, in general these rules are for tensors and we can say that any tensor may be decomposed as a sum of a symmetric and a skew symmetric tensor. So, matrix is a very special example illustration way of writing it.

Next what we will do? We will see that there are interesting quantities or parameters which are related to the angular velocity of the fluid. And based on that we will define a term called as vorticity.

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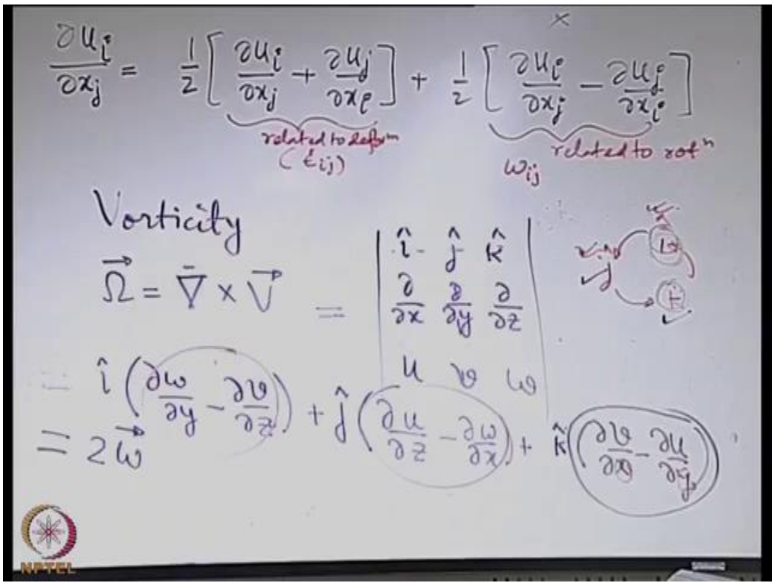

The entire subject of fluid mechanics in its fundamentals was developed by mathematicians. So, we are trying to give a physical basis or physical insight to whatever concepts that we are discussing. Because we have to keep in mind that after all we are trying to learn it in an engineering context and we have to understand that what physics goes behind these mathematical derivations.

At the same time when the subject was first developed it was developed in a sort of true mathematical way. And in certain cases quite a bit abstracted from any physical reality and in that perspective, the term vorticity which was defined as a vector, say it was defined as the curl of the velocity vector $\vec{\Omega} = \overline{\nabla} \times \vec{V}$.

But important thing is that such quantities are always defined in a general mathematical theory known as field theory. See field theory is something which is so general that it is applicable equally in electro magnetics than in fluid mechanics. And so, field theory talks about a vector field or maybe a scalar field, but in general a vector field. And what are the different rules that govern the behavior of the change of vector in a vector field.

So, when you are talking about electro magnetics you are having certain parameters when you are talking about fluid mechanics you are having different sets of parameters. These are physically different sets of parameters, but when you look into the mathematical field theory there is hardly any distinction between fluid mechanics and electro magnetics in terms of the basic mathematical theory that goes behind. We will see later on a couple of examples where you will find it very much analogous to as if there is an electrical field and something is happening with the electrical field.

Now, this vorticity when it is defined in this way. So, let us try to expand it in terms of its Cartesian components. Let us see and that will give us a fair idea of what physically it tries to represent. $\vec{\Omega} = \overline{\nabla} \times \vec{V} = \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{matrix}$

$$-\hat{i}\,(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) + \hat{j}\,(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$= 2\vec{w}$$

So, vorticity is a mathematical definition, but what it gives physically; a sense of rotationality in a flow. So, if this rotationality in the flow is very strong we say a vortex is created in the flow and the strength of that is given by the vorticity vector. So, that is the physical meaning of vorticity vector.

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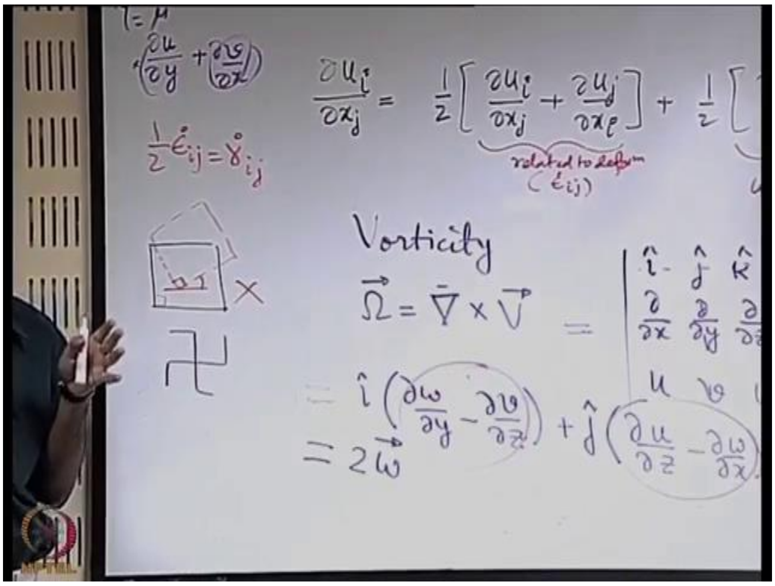

So, if you have say a line element say; this type of a symbolic line element you put it in a fluid. If the fluid has an element of rotationality it will change its orientation, otherwise it will move parallel to itself in the flow. So, that will give a visual understanding of whether the flow has an element of rotationality or not and that mathematically is quantified by the vorticity vector.

If you see it is if you know one of the components of the vorticity vector; it is possible to generate the other one intuitively without going into all the cross products.

In summary we have discussed the quantification of the linear and angular motion of the fluid elements.

Now, based on these we will come up with a few important conceptual understandings. So, those conceptual understandings are based on two important terminologies in fluid mechanics; one is stream function another is velocity potential let us see what is stream function.

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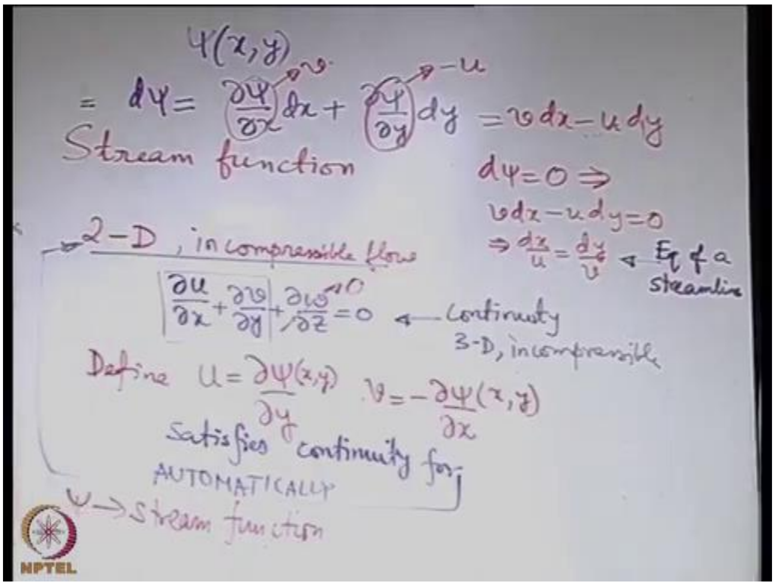

So, stream function is defined as follows to look into the definition let us first figure out that what is the motivation behind defining such a function.

Let us take an example of 2 dimensional incompressible flow and write the form of the continuity equation for that. What is the form of the continuity equation for 2 dimensional incompressible flow? If it was a general 3 dimensional flow; then it would have been $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\cancelto{0}{\frac{\partial w}{\partial z}}=0$.

Now when it is 2 dimensional; that means, the third velocity component is not important; so you are left with these 2 terms. Now, let us say that you are interested to find out the velocity components u and v. We will later on see that this is not the only equation that governs the change in u and v; there are other differential equations which need to be coupled.

But just in a notional or a mathematical form if instead of the 2 variables u and v; you could transform into a single parameter that is expressed u and v in terms of a single parameter, that satisfies automatically this form of the continuity equation; then u and v may be parametrized with respect to that new function.

Let us say we define $u = \frac{\partial \psi(x,y)}{\partial y}, v = \frac{\partial \psi(x,y)}{\partial x}$

so the objective is that we are trying to define the velocity components in terms of mathematical function which ensures the satisfaction of continuity equation automatically.

Because no matter how complex or how simple the flow is it should satisfy the continuity equation. So, this definition cannot violate that see why we are restricted to such a case because for a more general case it is not easy to find such parameter. So, if you had a 3 dimensional case; then you have not been possible to find out such a parameter automatically that satisfies this general equation.

So, this definition is restricted for a 2 dimensional incompressible flow. This function ψ is known as stream function. The speciality is that it has some relationship with the concept of stream line that we have learnt.

$$\psi(x,y)$$

$$d\psi = \cancel{\frac{\partial \psi}{\partial x}}^{v} dx + \cancel{\frac{\partial \psi}{\partial y}}^{*u} dy = vdx - udy$$

$$d\psi = 0 \Rightarrow vdx - udy = 0$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v}$$

Important conclusion is along a streamline there is no variation in stream function; that means, one stream line represents a particular constant stream function, that is the stream function equal to constant along a given stream line.

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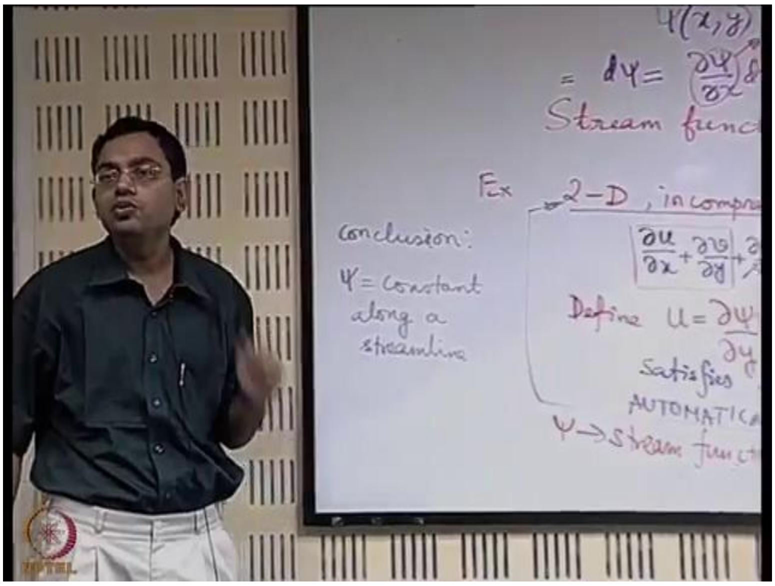

So, the conclusion is ψ = constant along a streamline and that is why the name stream function.

So, this relationship is not for all types of flow; this is only for that case when both are there streamline is always there, but always ψ = constant along a streamline is not relevant because always ψ is not defined; only for 2 dimensional incompressible flow these definition works and only for that case we may say that it is constant along a streamline. It does not mean that the streamline is not there if it is not a 2 dimensional incompressible flow it is very much there, but the stream function is not defined in this way. So, you cannot have a analogy or relationship between those 2.