Introduction to Fluid Mechanics
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Lecture – 25
Derivation of continuity equation

We were discussing about the continuity equation last time and we will see now that there may be a different way of deriving the continuity equation. Not only one different way, but there could be many different ways of looking into that. We will look into one such alternative way of deriving the continuity equation here and in our subsequent chapters we will look into other possibilities. So, more number of different ways, we look into it gives us a better and better insight of what is there actually in the continuity equation

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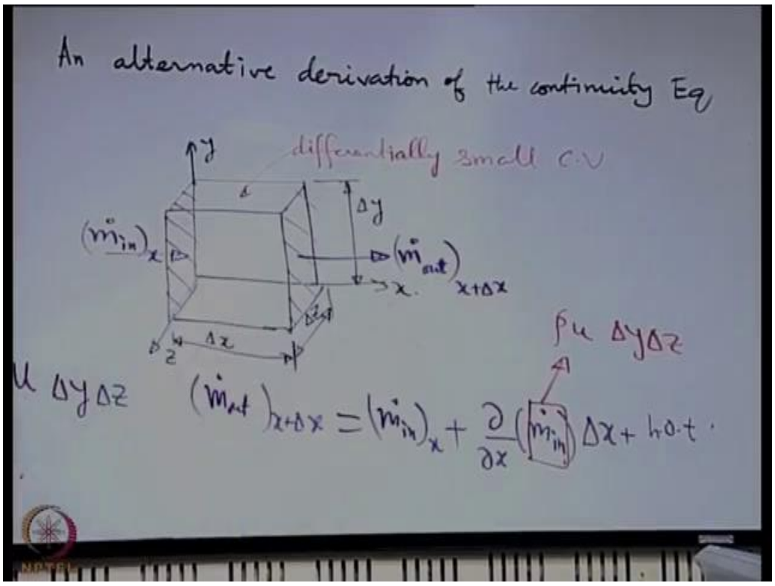

So, we look into an alternative derivation of the continuity equation. In this alternative derivation, our objective will be to look into the entire thing from Eulerean viewpoint; that means, we will identify a specified region in space across which fluid is flowing and that we call as a control volume. So, in a control volume of a particular extent, let us for simplicity in deriving the equations, assume that the control volume is of a rectangular parallelepiped shape. So, it has it is dimensions along x, y and z as say Δx, Δy and Δz which are small. And in the limit, we will take all these as tending to 0.

So, this is a differentially small control volume that is the entity. Now, what is happening across this control volume? Some fluid is coming in, some fluid is going out and that is occurring over six different faces and each face has a direction normal and basically the mass flow rate across that face is taking place normal to the direction of that respective face. So, if we consider the flow rate along x, then we should be bothered about which faces? We should be bothered about these two faces because these two faces have direction normals along x.

Let us see what is the mass flow rate that gets transported across these faces; so, there is a mass flow rate that enters the control volume along x. Across the opposite phase say there is some mass flow rate that goes out and that occurs at x + Δ x. Say, there is some mass flow rate coming at the rate of 10 kg per second and say there is a mass flow rate that leaves the control volume the rate of 8 kg per second. So, what the remaining 2 kg per second will do? That will increase the mass of the mass within the control volume see control volume has a particular volume. It does not have a fixed mass. So, if it is a compressible flow say, it is highly possible that the mass inside these changes. So, that remaining 2 kg per second may contribute to the rate of change of mass within the control volume.

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t}(m_{CV})$$

We use a partial derivative here because, by specifying the control volume here by some fixed coordinates, we are assuming that we are freezing it is locations with respect to position and trying to see what happens in that frozen position with respect to time. So, that is why a partial derivative with respect to time. So, when you say $\dot{m}_{in} - \dot{m}_{out}$, you have to remember that it has like contributions for flow along x, y and z.

So, to calculate the mass flow rate you require first to obtain the volume flow rate.

$u\Delta y\Delta z$ is the volume flow rate and $(\dot{m}_{in})_x = \rho u\Delta y\Delta z$

$$(\dot{m}_{out})_{x+\Delta x} = (\dot{m}_{in})_x + \frac{\partial}{\partial x}\left(\dot{m}_{in}\right)\Delta x + h.o.t. = (\dot{m}_{in})_x + \frac{\partial}{\partial x}(\rho u\Delta y\Delta z)$$

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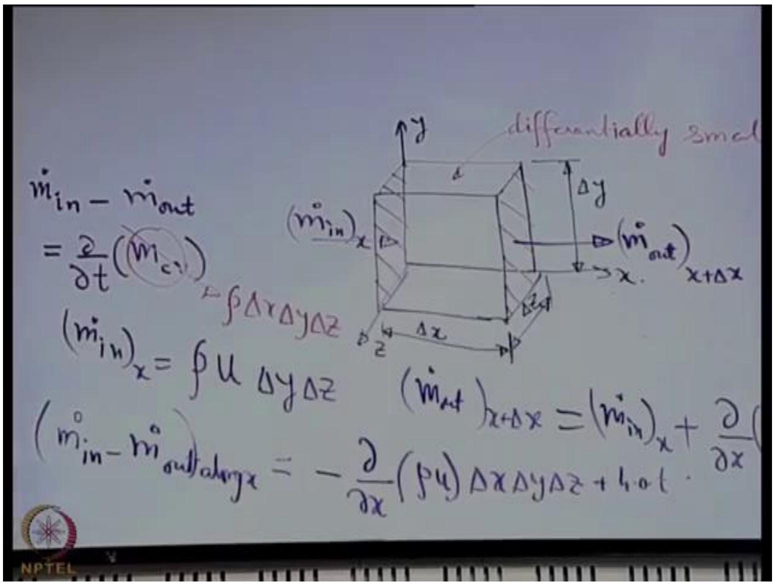

$$(\dot{m}_{in} - \dot{m}_{out})_{along\ x} = -\frac{\partial}{\partial x}(\rho u)\Delta x \Delta y \Delta z + h.o.t.$$

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t}(m_{CV}) = \rho \Delta x \Delta y \Delta z$$

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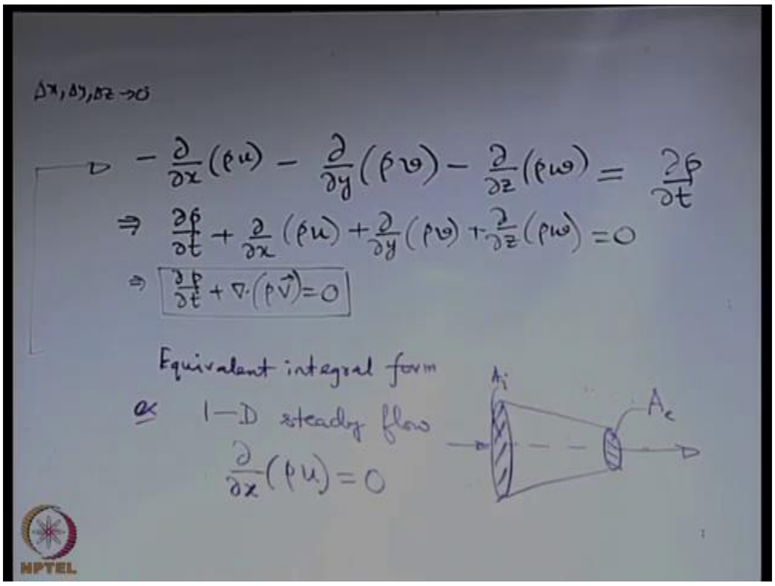

So, with that limit, that is Δ x, Δ y, $\Delta z \rightarrow 0$

$$-\frac{\partial}{\partial x}(\rho u)-\frac{\partial}{\partial y}(\rho v)-\frac{\partial}{\partial z}(\rho w)=\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

$$\frac{\partial \rho}{\partial t}+\nabla(p.\vec{v})=0$$

So, we have seen at least two different ways of deriving the continuity equation and keep in mind, these are not the only two ways, but at least these have given us some insight of what this law or what this equation is talking about. We will have a couple of important observations related to this before we go on to a problem where we illustrate how to make use of such equations. So, the first point is that you see this is a differential form.

Now, let us say that you want to express in a integral form. So, how will you do it? We will later on formally see one methodology by which you can convert easily from a differential to an integral form. But without going into that formality, let us look into a very simple example by which we see that how to do it. So, we are now interested about equivalent integral form. We will not go for the most general case. That we will study later, but we will consider a very simple case as an example; 1-D steady flow, say the flow is taking place along x.

May be just for your visual understanding, let us say that the flow is taking place through a nozzle like this; a nozzle something where you have when the flow is entering, it is entering with a higher velocity and as it is moving along it the area of cross section gets reduced. So, the velocity of the flow gets increased. So, you may assume that it is something like a conical shape maybe frustum of a cone or something like that. The shape is not important for us. We will just keep in mind that there is an inlet section with area A_i and there is an outlet section with our exit section with area A_e and the flow is taking place along x.

When you have one dimensional flow, $\frac{\partial}{\partial x}(\rho u)=0$

Now, what we will do is, we will try to integrate this over the entire volume of the nozzle.

$$\int_{\forall}\frac{\partial}{\partial x}(\rho u)d\forall=0$$

Why we require a small volume element to consider because u is spatially varying. So, we are taking you at a location where you that you is that u at that particular x and then we are integrating that over the entire volume by considering such elemental volumes. So, that is that integrated over the entire volume that should be equal to 0. It is very straightforward. If the function is 0, it is integral should be 0. On the other hand, if the integral is 0, function need not always be 0, right. But we will see later on that there are certain cases when if the integral is 0.

We may say that the function itself is 0 under certain important considerations, but not for all considerations, but here it is the other way which is more straightforward. Now, when you have it see our objective is to convert this volume integral to an area integral because we are interested about the areas which the fluid is flowing.

$$\int_{\forall} (\nabla.\vec{F})d\forall = \int_{A} (\vec{F}.\hat{n})dA = \int_{A} (\vec{F}.d\vec{A})$$

$\vec{F} = \rho u\hat{i}$. So, the divergence of that will give you this partial derivative

$$\int_{\forall} \frac{\partial}{\partial x}(\rho u)d\forall = 0 \Rightarrow \int_{\forall} \nabla(\rho u\hat{i})d\forall = 0$$

Now, if you look at this theorem, this is the mathematical statement of the theorem. What it has is a very important understanding. What is this A and what is this V? It is not any arbitrary A and arbitrary V. This A is the area of the surface that bounds the volume V closely. So, when you have the volume V here, it is bounded by say lateral surfaces and this cross sections. So, when we are writing this in terms of an area integral, that area integral should consider A_i A_e and the lateral surfaces also lateral surfaces and then will not be important because there is no flow across those surfaces. So, those are like irrelevant from flow computation considerations. But fundamentally, it is the entire surface that is bounding the volume.

$$\int_{A} (\rho u\hat{i}.\hat{n})dA = 0$$

Now, let us look into this form. So, when you consider this dA that area element, now you have as we mentioned three types of like one is inlet, another is exit and another we may call as wall, right across which there is no flow. So, when you have the wall, it is not necessary to calculate to bother about this integral for the wall because there is no flow there. So, we will therefore, break it up into two integrals; one for the area A_i, another for the area A_e, other areas are not relevant.

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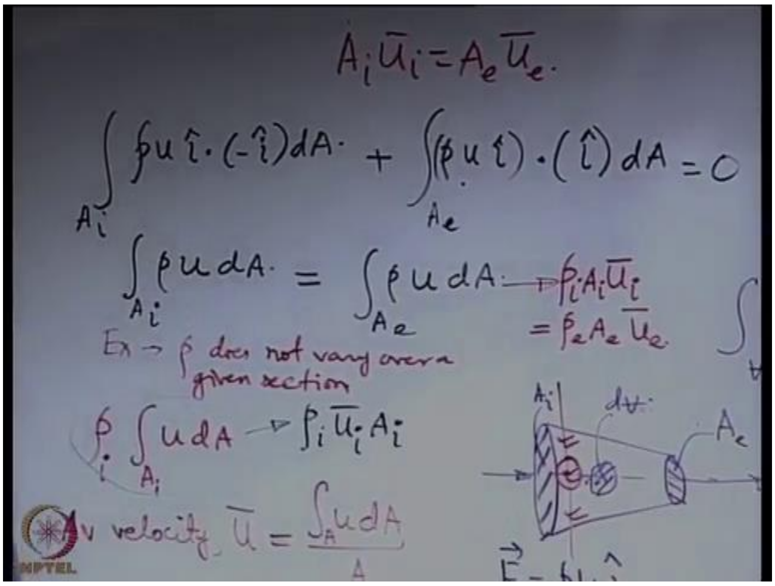

$$\int_{A_i} \rho u\hat{i}.(-\hat{i})dA + \int (\rho u\hat{i}).(\hat{i})dA = 0$$

$$\int_{A_i} \rho u dA = \int_{A_e} \rho u dA$$

At for steady, whatever is the mass flow rate across this section the same must be the mass flow rate out across this section. That is what it is physically saying mathematically the statement is straightforward. Now, it is many times convenient to express this in terms of the average velocity because it might so happen that u is a function of the transverse coordinates. Let us say, we have transverse coordinate as maybe say r or y, that type of a coordinate and it is possible and it is almost always likely that u will be the function of a transverse coordinate because u will be 0 at the walls by no slip condition. And then, u will change may be maximum at the center line. So, it is expected that along the transverse direction you will vary. So, it is not that we are talking about a cross sectionally constant u, it is rather a cross sectionally variable u.

Now, if we assume that ρ is not varying across, the section as an example. So, let us take an example where ρ does not vary over a given section but it may vary from one section to the other. So, ρ does not vary over a given section; that means, you can take that ρ out of the integral and $\rho\int_A udA$

Average velocity, $\bar{u} = \frac{\int_A udA}{A}$

Now, if you have the same volume flow rate with an equivalent velocity that would have been uniform throughout, then that uniform equivalent velocity is the average velocity. So, what we are basically doing, we are equating the volume flow rate. In one case, it is a variable velocity, the real case. In the other case, it is an equivalent idealistic case where it is a uniform velocity, but the end effect the mass the volume flow rate is the same. And then, that equivalent velocity equivalent uniform velocity over that section is known as the average velocity.

$$\rho_i \int_{A_i} udA \rightarrow \rho_i \bar{u}_i A_i$$

$$\int_{A_i} \rho udA = \int_{A_e} \rho udA \rightarrow \rho_i \bar{u}_i A_i = \rho_e \bar{u}_e A_e$$

$$\bar{u}_i A_i = \bar{u}_e A_e$$

So, what are the assumptions under which it is valid? See we have, of course, we may go on deeper and deeper into the assumptions, but let us talk about only the major assumptions what are the major assumptions; first ρ is a constant the density is a constant, then we are talking about these velocities not local velocity at a point, but cross sectionally average velocity. If it is an ideal fluid flow, then it is possible that local velocity is same everywhere because the velocity gradient is created by viscosity. So, if you have no viscous effect, then the effect of the wall is not propagated into the fluid and it is possible that there is a uniform velocity profile.

Now, next let us look into another issue that we have discussed about the continuity equation in a general vector form, but we have not looked into the other coordinate systems. We have looked into the Cartesian coordinate system, but let us say that we are also interested about the cylindrical polar coordinate system. It is important if you have something of say cylindrical symmetry some body of cylindrical symmetry.

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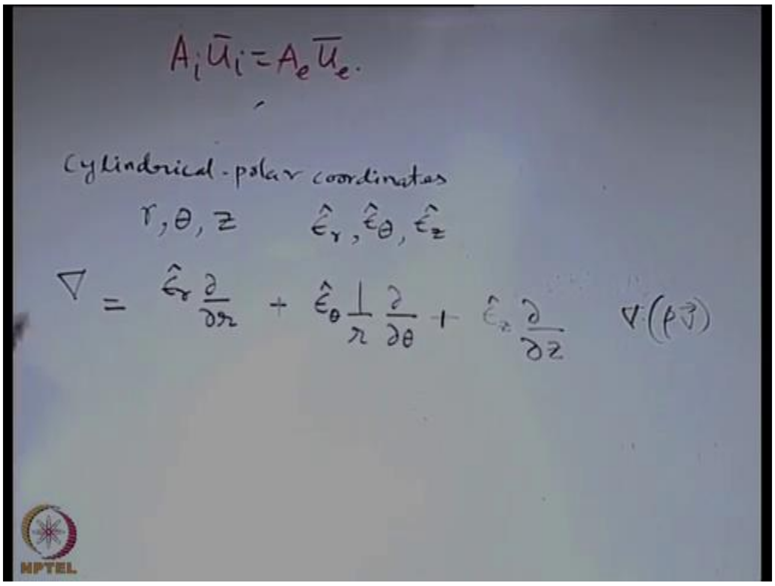

So, if you consider say cylindrical coordinates. So in the cylindrical polar coordinate, you have a polar nature; that means, you have the r θ coordinate system just like the polar coordinate. And you also have a three dimensionality. So, you have the axial coordinate system given by the z. So, r θ z coordinate system and let us say that these have their unit vectors as given by ε_r ε_θ and ε_z just like i, j, k. Now we have to see that what are the differences in the Cartesian system and in this system.

$$\nabla = \hat{\varepsilon}_r \frac{\partial}{\partial r} + \hat{\varepsilon}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varepsilon}_z \frac{\partial}{\partial z}$$

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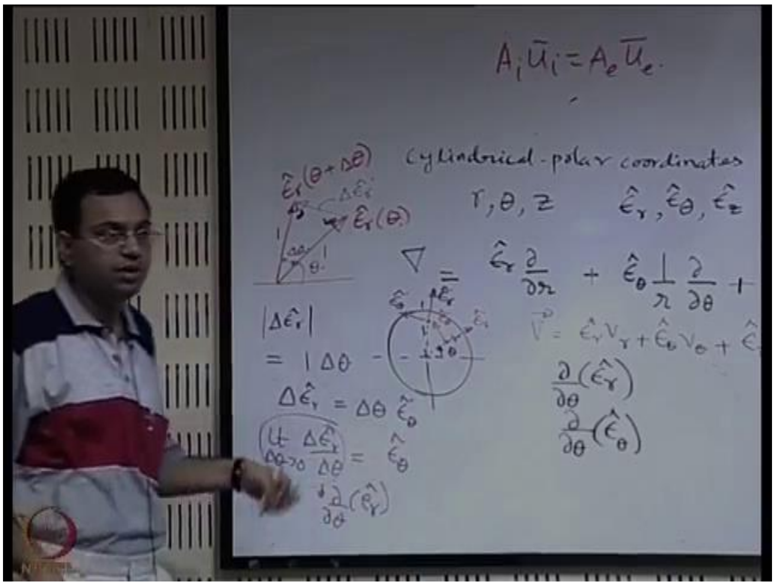

$$\vec{V} = \hat{\varepsilon}_r v_r + \hat{\varepsilon}_\theta v_\theta + \hat{\varepsilon}_z v_z$$

So, when you are differentiating, it is possible that you have to find out $\frac{\partial}{\partial \theta}(\hat{\varepsilon}_r)$ and $\frac{\partial}{\partial \theta}(\hat{\varepsilon}_\theta)$

So, we will find out one let us say you want to find out the derivative of ε_r . So, how do you look into that? Let us say that you have ε_r for a particular angle ε_θ as this one. And let us say that ε_r has changed with the angle $(\theta + d\theta)$.

$$|\Delta \hat{\varepsilon}_r| = 1.\Delta\theta$$

As $\Delta\theta \to 0$, sum of the three angles of this triangle is 180^0. This is an isosceles triangle. So, these two angles should be equal. So, when these tends to 0, these two tend to 90^0 almost.

$$\Delta \hat{\varepsilon}_r = \Delta\theta \hat{\varepsilon}_\theta$$

$$\underset{\Delta\theta \to 0}{Lt} \frac{\Delta \hat{\varepsilon}_x}{\Delta\theta} = \hat{\varepsilon}_\theta$$