Introduction to Fluid Mechanics
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Lecture – 24
Deformation of fluid elements – Part I

Next, what we will do? We will start analyzing the Deformation of fluid elements.

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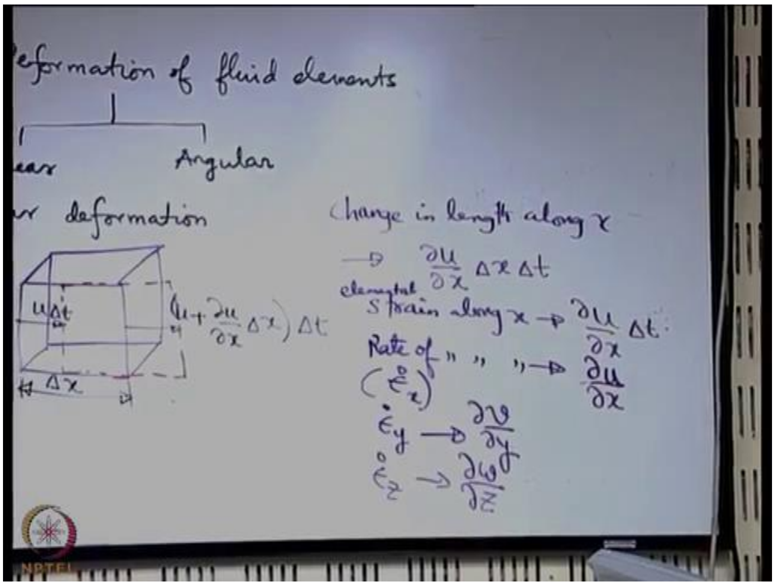

Why this is very important? Because we have seen that fluids are characterized by deformation, they undergo continuous deformation under the action of even a very small shear force and the relationship between the sheer force and the shear stress and the rate of deformation is something which is unique to the constitutive behavior of different fluids.

So, we must first understand that how to characterize deformation of fluid elements in terms of the velocity components. Once we understand that it will be possible for us to mathematically express different types of deformations in terms of the velocity components say u, v and w. When we do that we have to keep in mind that we will be essentially bothering about two types of deformations; one is the Linear, another is the Angular deformation.

When we talk about the linear deformation, it may eventually give rise to a change in volume of the fluid element also. Because if you have a length element and the length element gets changed; a volume element is comprising of several such length elements. So, if linear dimension gets change, the volume is also likely to get changed.

Initially, we will think of how we can estimate the linear deformation. So, we will start with the linear deformation. To understand or to get a visual feel, we will consider a fluid element like this; maybe we may consider even a 3 dimensional fluid element if you want. But that will not make the thing more complicated because at the end, we will be dealing with linear deformations in individual directions. See why we use a coordinate system for analyzing a problem?

The reason is like say when you think of xyz the Cartesian coordinate systems; these are independent coordinates. A combination of which describe the total effect in the system. So, when you are thinking of a linear deformation along x, you may be decoupled from what is the linear deformation along y and z and these individual effects you can superimpose because you are dealing with linearly independent components and these vector components actually give you linearly independent basis vectors like components along x, y or z.

So, similar concept whenever we are considering a change along x maybe we are bothered only with respect to like what is the change in the linear dimension along x disregarding what happens along y and z. So, let us keep that target let us say that Δx is the length of the fluid element which is originally there and now what is happening? Now we are having a change in time and because of a change in time, now you see that let us consider the front face of this cuboid.

So, this left phase over a time interval of Δt will traverse at this displacement; will undergo a displacement, what is the displacement? If u is the velocity at this location, simply u into Δt. We are considering the time interval Δt to be very small. So, it is like just a $u\Delta t$. The right phase will also undergo some displacement; what is that? So, if this is the x direction, the new u here is same is not the same as the u at the left phase, but this is because of change in u due to change in x.

So, this is $\left(u+\frac{\partial u}{\partial x}\Delta x\right)\Delta t$. So, if you consider only the front phase and only subjected to this motion, say we freeze all other events just for a clear picture. So, maybe now it is having a new configuration shown by this dotted line. So, what is the change in its length along x? That is the final length minus the original length.

So, what is the final length? So, what is the net change? See the right hand phase has got displaced by this amount; the left hand phase has got displaced by this amount. So, the net displacement is the difference between these two. So, what is that change in length? So, $\frac{\partial u}{\partial x}\Delta x$ Δt is the change in length along x. What is therefore, the strain along x; the change in length per unit length?

So, the strain along x this is the elemental strain because we have considered only a small part of the fluid which is having an extent of Δ x. So, this is elemental strain along x. As we have discussed earlier, we are not just interested about the strain for a fluid because if you allow it to grow in time the strain will be more. See if this Δ t is larger and larger and you integrate it over a large interval of time this will be trivially more and more.

So, measuring strain in a fluid is nothing that is important. It is just a function of the time that has elapsed; what is more important is the rate of deformation or the rate of strain. So, the rate of strain along x; what is that? It is basically $\frac{\partial u}{\partial x}$.

$$\dot{\varepsilon}_x \to \frac{\partial u}{\partial x}, \dot{\varepsilon}_y \to \frac{\partial u}{\partial y}, \dot{\varepsilon}_z \to \frac{\partial u}{\partial z}$$

So, we have been successful in finding out a very simple thing, what is the rate of linear deformation along x, y and z in terms of the velocity components.

So, if you are given u as a function of position; v as a function of position and w as a function of position, by simple partial differentiation, it will be possible to find out the rates of change. Now we are interested not only just in terms of the rate of change in the linear dimension, but maybe rate of change in the volume. So, to understand that what is the rate of change in the volume, let us say that we are having this fluid element which has dimensions along x y z as Δ x Δ y and Δ z.

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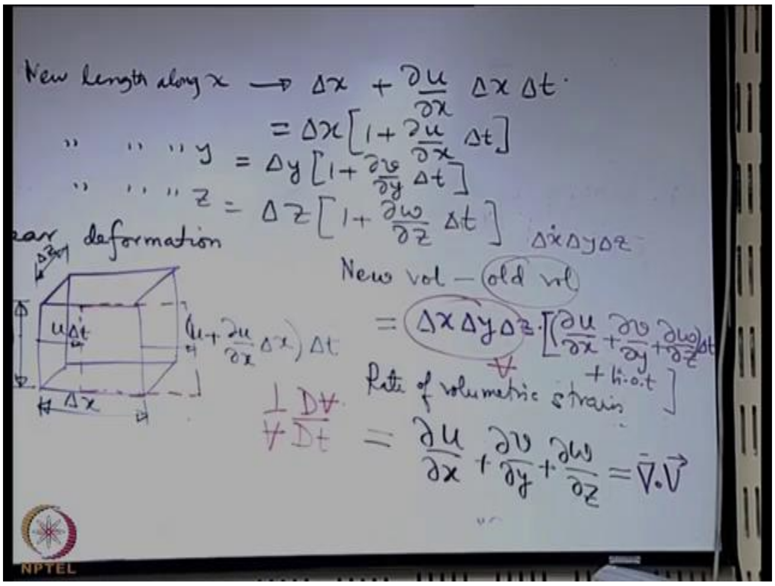

So, we set up coordinate axis as this is x, this is y and this is z. So, Δ x Δ y and Δ z. Now what is the new length? So, to we are interested to get the new volume. So, what is the new volume?

$$\text{New length along x} \to \Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t = \Delta x\left(1 + \frac{\partial u}{\partial x}\Delta t\right)$$

$$\text{New length along y} \to \Delta y + \frac{\partial u}{\partial y}\Delta y \Delta t = \Delta y\left(1 + \frac{\partial u}{\partial y}\Delta t\right)$$

$$\text{New length along z} \to \Delta z + \frac{\partial u}{\partial z}\Delta z \Delta t = \Delta z\left(1 + \frac{\partial u}{\partial z}\Delta t\right)$$

$$\text{New volume } - \text{ Old Volume} = \Delta x \Delta y \Delta z\left[\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\Delta t + h.o.t\right]$$

So, when you say find out per unit volume, you are basically dividing it by this Δ x Δ y Δ z. So, this is like the original volume. Let us give it a symbol $\forall$. So, that is the original volume. So, what is the rate of volumetric strain? This change in volume divided by volume divided by Δ t and take the limit as Δ t tends to 0, when the all other higher order terms in the limit will be 0. So, it is not that we are neglecting. The one Δ t here will remain even after division by Δ t that will be tending to 0 as in the limit Δ t tends to 0.

So, we may write it in terms of the total derivative. See the volumetric strain; it may be due to many things change in time change in position and a combination. So, we are not bothered about that what is the individual effect; we are bothered about the total effect. What is the net change in the fluid element volume because of this? So, this should be expressible in terms of the total derivative.

Rate of volumetric strain $\frac{1}{\forall}\frac{D\forall}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \bar{\nabla}.\vec{\nabla}$

This leads to a very important definition, the definition is with regard to incompressible flow. So, when we say that a flow is incompressible.

So, incompressible flow by the name it is clear that we are looking for a case when the fluid element does not change in volume.

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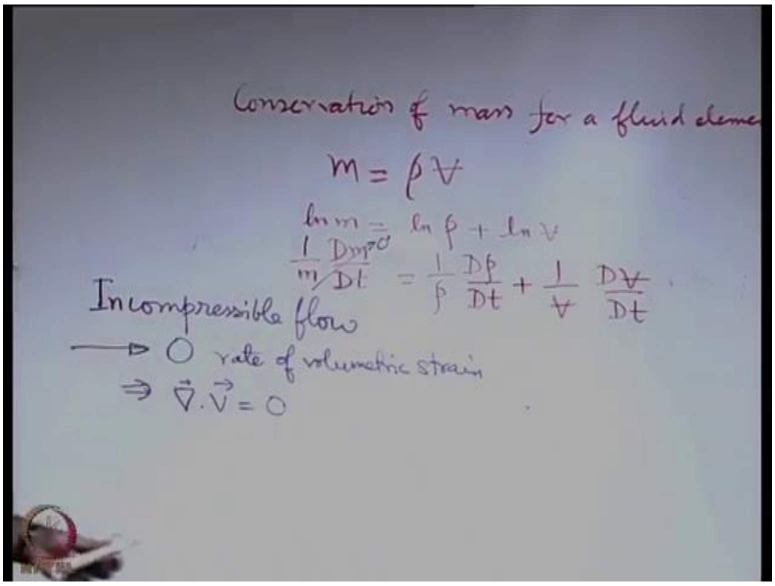

So, incompressible flow, it will have what signature? One and only important signature 0 rate of volumetric strain because the fluid element may not be changing its volume that is the meaning of that is even the literal meaning of incompressible that you cannot really compress it.

So, 0 rate of volumetric strain and that boils down to the divergence of the velocity vector is equal to 0. So, if you are given a velocity field and you are asked to check whether it is compressible or incompressible flow, then it is it is it is possible to check by looking into the fact whether it is satisfying this equation or not. If it satisfies this equation, we say that it is an incompressible flow. Keep in mind the distinction between this definition and incompressible fluid definition.

So, earlier we also introduced the concept of incompressible fluid and we said that a fluid is incompressible, if its density does not significantly change with change in pressure. So, that is incompressible fluid. Now we are talking about incompressible flow. And these two are again related, but different concepts that we have to keep in mind. So, when you are having an incompressible flow, it is possible to characterize the particular flow in terms of its mechanism by which it satisfies the overall conservation of mass.

To understand how it does? Let us try to write an expression for conservation of mass of the fluid element. So, we will now write conservation of mass for fluid. For a fluid element let us say that m is the mass of a fluid element. You can express it in terms of the density and the volume. Let us say that ρ is the density and $\mathcal{V}$ is the volume.

Since the mass is conserved of a fluid element. So, there will be 0 rate of change of mass. So, the since we know already the expression for the volumetric strain and in that volumetric strain 1 by v appears; it may be useful to utilize that expression by taking log of both sides and then differentiating. Because then $\frac{1}{\mathcal{V}}$ will automatically come out.

So, let us take the log of both sides and then differentiate with respect to time. When we say we want to differentiate with respect to time, it has to be a total derivative. So, because it is a fluid element now, it may have change with respect to change in position, time whatever we are bothered about. Now, the total effect because the conservation of mass is not for individual effects, it is a combination of total effects that gives rise to a mass of a fluid element is conserved.

So, when we write say when we differentiate it with respect to time by keeping that in mind, we have the left hand side like this which again becomes 0 because the mass of the fluid element is conserved.

$$m = \rho \mathcal{V}$$
$$\ln m = \ln \rho + \ln \mathcal{V}$$
$$\frac{1}{m}\frac{Dm}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt}$$

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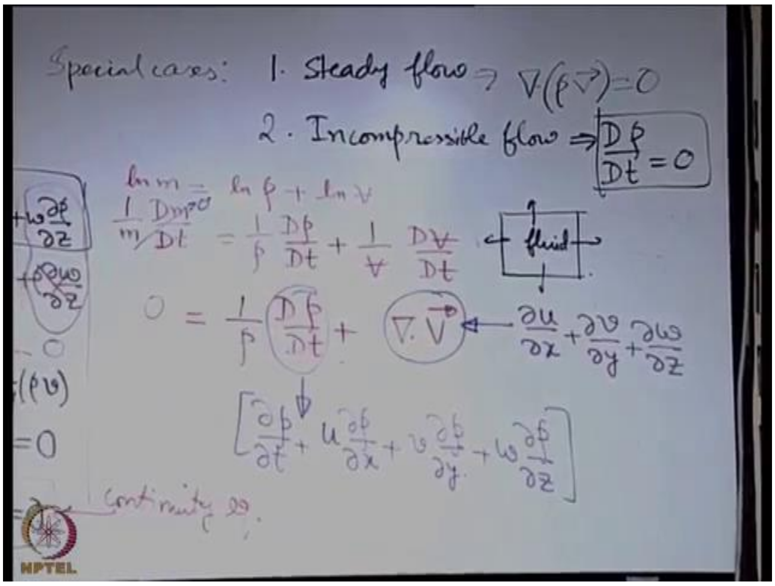

$$0=\frac{1}{\rho}\frac{D\rho}{Dt}+\nabla.\vec{\nabla} \text{ where } \frac{D\rho}{Dt}=\frac{d\rho}{dt}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z} \text{ and } \nabla.\vec{\nabla}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

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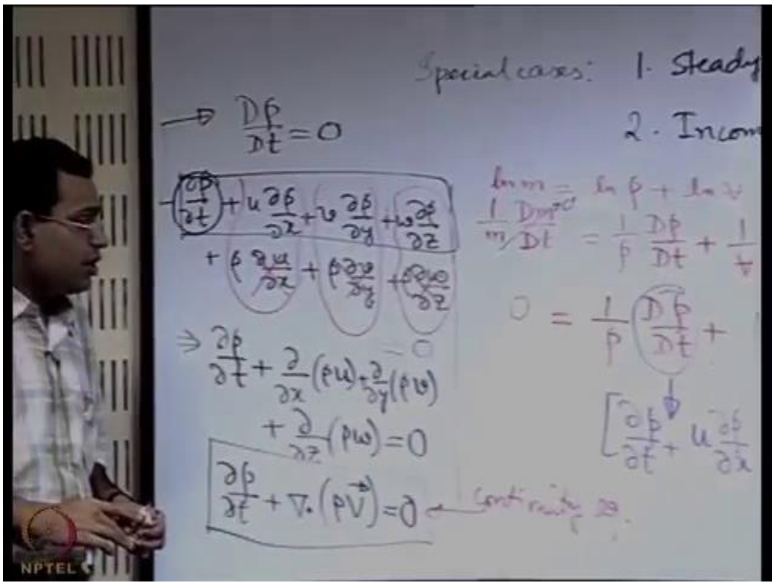

$$\frac{d\rho}{dt}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial u}{\partial y}+\rho\frac{\partial u}{\partial z}=0$$

$$\Rightarrow\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{V})=0$$

This equation in a general understanding is supposed to be the most fundamental differential equation in fluid mechanics because no matter how complex or how simple the flow field is, it should satisfy the law of conservation of mass.

So, this is a differential equation expressing the law of conservation of mass for a fluid element and this is known as continuity equation. So, if you are given a velocity field, you must first check whether it is satisfying the continuity equation. If it does not satisfy the continuity equation, it is an absurd velocity field. It may be mathematically something, but it does not physically make any sense because it has to satisfy the mass conservation.

Now, briefly let us look into certain special cases of these. So, what are the special cases? The first special case we consider as steady flow. So, when you consider a steady flow, then how this equation gets simplified to? So, steady flow means the first term. At a given remember what is the definition of steady? At a given position any fluid property will not change with time.

Steady flow $\rightarrow \nabla.(\rho\vec{V}) = 0$

Let us consider a second case incompressible flow. We have to keep in mind that there is a very big miss concept that we should try to avoid; what is that? Many times we loosely say incompressible means density is a constant. It is a special case of incompressible flow, but it is not a general case of incompressible flow because general case of incompressible flow is what? The divergence of velocity vector equal to 0; that is the definition.

Now, where does it ensure that ρ is a constant that basic definition never ensures that ρ is a constant. At the same time it can be shown that if ρ is a constant, then this will be satisfied. So, the converse is true; that means, ρ is a constant is a special case of incompressible flow, but it is not a general case. What is the general case, let us look into that. So, when you are looking about the general case, you have to see the continuity equation. So, if when you look at the continuity equation, look into this primitive form that is not the compact form, but this form before that.

Incompressible flow $\rightarrow \frac{D\rho}{Dt} = 0$

See a very interesting thing, it does not mean that ρ is a constant because ρ might be a function of position and time in such a way that this collection of terms eventually gives rise to 0. If ρ is a constant this collection will definitely give you a give it to be 0, but that is a trivial solution that is a trivial solution to the case that is ρ is a constant; therefore, any derivative of ρ with respect to time or position is 0.

But even if any derivative of ρ with respect to position and time is not 0, still the net effect may be 0 and then even though ρ is a variable, we will say that the flow is incompressible. So, a variable density flow may also be an incompressible flow; this is a very important concept. So, incompressible flow need not always be a constant density flow.

So, a typical example is let us say that you have a domain like this; within this there is a fluid. Now this fluid change at it is phase say it was in a particular phase, it was in liquid phase; now it becomes a vapor phase. So, when it becomes vapor phase, it becomes lighter. So, the same mass now cannot occupy this volume.

Some extra mass will leave because you are constraining the volume and if you are having a change in density, you must have a flow to accommodate a change in density so that whatever fluid is there now is active accommodated within the volume that was given to you. So, you can see that you might have a change in density at a fixed position with time because maybe with time the phase change has triggered. So, with time the density has changed. So, this has to be now adjusted with some u, v, w so that the net effect may still be 0.

So, it might so happen that now here the net effect it may be 0, it may not be 0. So, let us take an example where the net effect is 0. What is that example? Maybe there was a fluid now it is getting frozen and because of freezing its volume gets change. So, it is possible. So, its density has got changed, but we do not call say liquids or solids as compressible fluids. So, what has happened because if with freezing there is a shrinkage, then there will be a deficit in volume here; maybe to satisfy the deficit in volume, there might be a material supply from all sides.

So, it is possible that to make a balance of what is happening locally and what is what is happening over the volume element, you might to adjust these things with velocity across the different phases of the element. So, in summary we can say that incompressible flow definition is the total derivative of ρ with respect to t_0, but not just ρ is a constant.