Introduction to Fluid Mechanics
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Lecture – 23
Acceleration of fluid flow

We were discussing about different flow visualization lines in our last class and we will discuss one more flow visualization line which is called as timeline.

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So, what is the timeline? If you have a snapshot at a particular time in the flow field where you mark nearby particles. So, nearby fluid particles which are located in the flow field at a given instant of time, if you somehow mark those particles by some way then if you now get the snapshot at different times, it will give a picture of evolution of the flow field as a function of time and that is known as a timeline. So, it is nothing, but like snapshot of nearby fluid particles at a given instant of time, that is called as a timeline. So, let us look into small movie to see that what we mean by a timeline.

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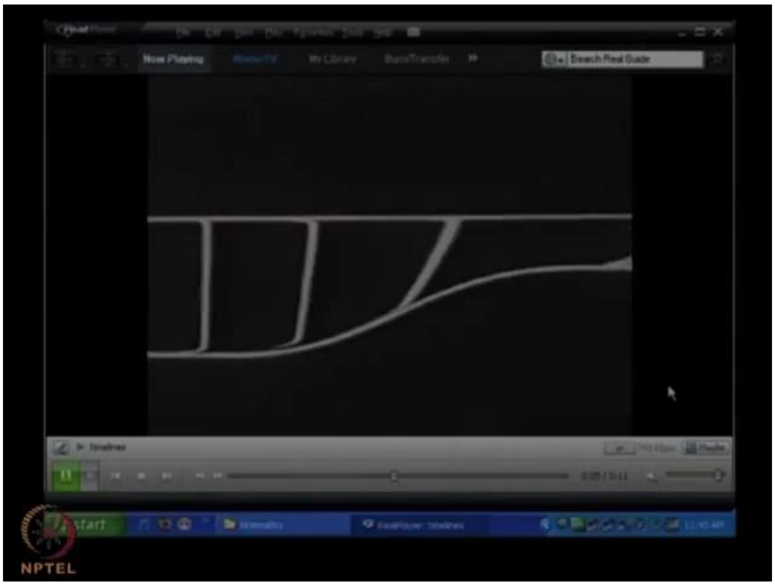

So, if you see now this gives snapshot at different instants of time of nearby fluid particles. And, in a way it gives a sense of the velocity profiles at different instants of time. You can see that in this example the flow passage is narrowing and as the flow passage is narrowing the fluid is moving faster to make sure that the mass flow rate is conserved. We will see later on that formally this is described by the continuity equation and in maybe a differential form or an integral form, but at least this gives us a visual idea of what the time line is all about.

Now, with this background on the flow visualization lines we have now understood that how we can visualize the fluid flow in terms of some imaginary description like through the stream line, streak line, path line or maybe the time line. Next we will go into the description of acceleration of fluid flow. So, we have discussed about the velocity the next target is the acceleration.

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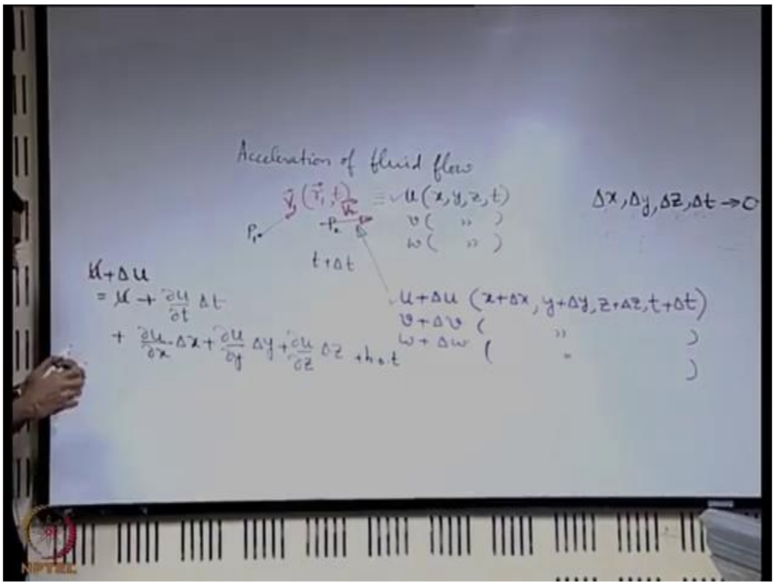

Let us say that you have a fluid particle located at a position P at specifically the location P_1 at time equal to t. And how the velocity is described here? The velocity described is described here through a velocity vector V which is a function of r_1, that is the position vector of the point P_1 and the time t this is nothing, but the Eulerian description.

$$V(\vec{r},t) = \begin{matrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{matrix}$$

So, if you are using a Cartesian coordinate system, 3 independent coordinates, space coordinates plus time coordinate that together give the velocity at a particular point. So, if the fluid particle is located at P_1 the velocity at that point is basically the velocity of a fluid particle located at that point and that is given by these components. Now, let us say that at a time of $t + \Delta t$ these things get changed.

Now, at a time $t + \Delta t$ what happens, this fluid particle is no more located at this point the fluid particle is located at a different point. So, let us say that the fluid particle is located at a point P_2. So, at the point P_2 now let us say that the velocity is whatever at some arbitrary velocity. So, initially it may be velocity at the point 1 say V_1 now it is V_2 which is again a function of its local position and time.

$$\vec{V}_2 = \begin{matrix} u + \Delta u(x + \Delta x, y + \Delta y, z + \Delta z) \\ v + \Delta v(x + \Delta x, y + \Delta y, z + \Delta z) \\ w + \Delta w(x + \Delta x, y + \Delta y, z + \Delta z) \end{matrix}$$

So, we can clearly see that there is an original velocity in terms of its 3 components, there is a change velocity in terms of its 3 components. And, if we want to find out the acceleration see the basic definition of acceleration is based on a Lagrangian reference frame, that is the rate of time rate of change of velocity in a Lagrangian frame not in an Eulerian frame.

All the basic definitions in Newtonian mechanics that we have learnt earlier are based on Lagrangian mechanics. So, when you say that it is a rate of time rate of change of velocity then that has to deal with the time rate of change of velocity of maybe an identified fluid particle which earlier was at P_1 now is at P_2. So, if we want to find out the change.

So, you can write of course, you can write it in terms of the 3 different components, but just first implicitly let us just write for the x component. Similar things will be there for y and z component.

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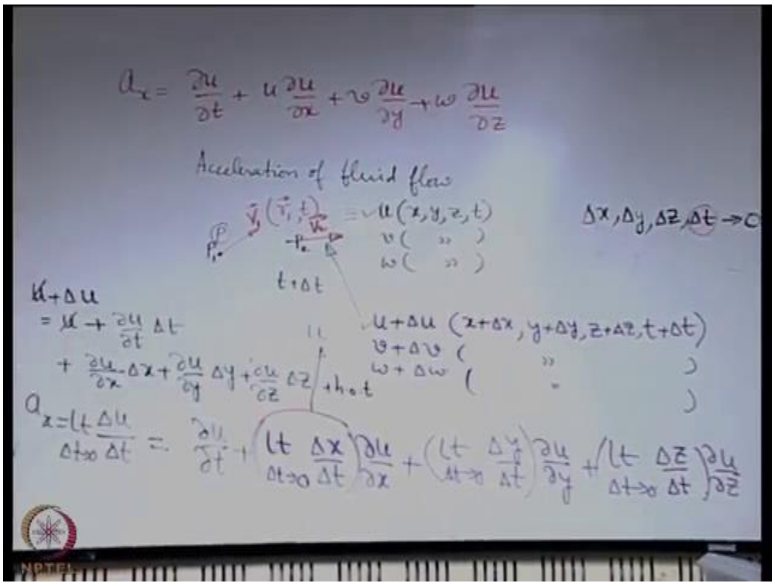

So, how can you write $u + \Delta u$ as a function of u. So, $u + \Delta u$ is now dependent on the local position of the particle and the time that has elapsed. So, it is a function of it depends on what; it depends on the original u plus the change. So, what was the original u? That was u plus see it is a function of 4 variables. So, you again it is a same mathematical problem that there is a function of 4 variables it is known at a given condition. Now, you make a small change in each of these variables and you want to find out the new function. Again you can express it through a Taylor series expansion; now it is a function of multiple variables instead of a single variable.

$$u + \Delta u = u + \frac{\partial u}{\partial t}\Delta t + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z \text{ +h.o.t}$$

These we have just written the first order term in the Taylor series. Since, it is a function of 4 variables you have 4 first order derivative terms. Similarly, you will be getting second order derivative terms and so on. But, we will neglect the higher order terms by considering that these $\Delta x, \Delta y, \Delta z, \Delta t$ are very small.

So, we have to keep in mind that all $\Delta x, \Delta y, \Delta z, \Delta t \rightarrow 0$

$$a_x = \underset{\Delta t \rightarrow 0}{Lt} \frac{\Delta u}{\Delta t}$$

So, when you do that basically, what we are doing we are dividing the left hand side by Δt. So, right hand side is also divided by Δt and the limit is taken as $\Delta t \rightarrow 0$. So, the first term is straightforward; let us look into the next terms.

$$a_x = \underset{\Delta t \rightarrow 0}{Lt} \frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial t} + \left(\underset{\Delta t \rightarrow 0}{Lt} \frac{\Delta x}{\Delta t} \right)\frac{\partial u}{\partial x} + \left(\underset{\Delta t \rightarrow 0}{Lt} \frac{\Delta y}{\Delta t} \right)\frac{\partial u}{\partial y} + \left(\underset{\Delta t \rightarrow 0}{Lt} \frac{\Delta z}{\Delta t} \right)\frac{\partial u}{\partial z}$$

And, these terms are basically representatives of that we will formally see that how they represent such a situation. So, now let us concentrate on these limiting terms; say the first limiting term. What it is representing? It is representing the time rate of change of displacement along x of the fluid particle over the period Δ t.

Now, you have to keep in mind that we are thinking about the limit as Δ t tends to 0 this is a very important thing. What is the significance of this limit as $\Delta t \rightarrow 0$? When $\Delta t \rightarrow 0$ P_1 and P_2 are almost coincident right; that means, let us say that P_1 P_2 all those converge to some point P. And, that point is a point at which say we are focusing our attention to find out what is the change of velocity that is taking place.

So, when in the limit $\Delta t \to 0$ we are considering the Eulerian and Lagrangian descriptions margin; this is very very important. So, we are trying to see what is our motivation, we know something and we are trying to express something in terms of what we know. What we know we know the straightforward Lagrangian description of acceleration.

We are trying to extrapolate that with respect to an Eulerian frame. To do that we must have an Eulerian Lagrangian transformation and essentially we are trying to achieve that transformation in a very simple way; that as the $\Delta t \to 0$ Eulerian and Lagrangian description should coincide.

And, then what does it represent? It represents the instantaneous velocity x component of the instantaneous velocity of the fluid particle located at P; that means, it represents the x component of the fluid particle located at P. Since you are focusing on attention on P itself and the velocity of the fluid particle if it is neutrally buoyant is same as the velocity of flow; we can write that this is same as u at the point P.

See writing this as u is very straightforward, understanding it conceptually is not that trivial and straightforward. If the Eulerian and Lagrangian descriptions did not merge, we could not have been able to write this because, this is on the basis of a Lagrangian description and this is Eulerian velocity field. How these two can be same? They can be same only when we are considering a particular case when in the limit as $\Delta t \to 0$. So, wherever we are focusing our attention at that particular point this represents the velocity of the fluid particle. If the fluid particle is neutrally buoyant with the flow then it is like an inert stress or particle moving with the flow.

And, then it would have the same velocity as that of the flow at that point, at that point at that instant. However, if the fluid particle has a different density than that of the flow then this would be u of the fluid particle. So, fundamentally this is u of the fluid particle not u of the flow field. If it is neutrally buoyant then it becomes same as u of the flow field. If it is an inert tracer particle in the flow which is the definition of the fluid particle then it is definitely same as u at that point.

But, if it is a fluid if it is a particle of a different characteristic different density characteristic than that of the flow, it may be different from that of the velocity field at that point.

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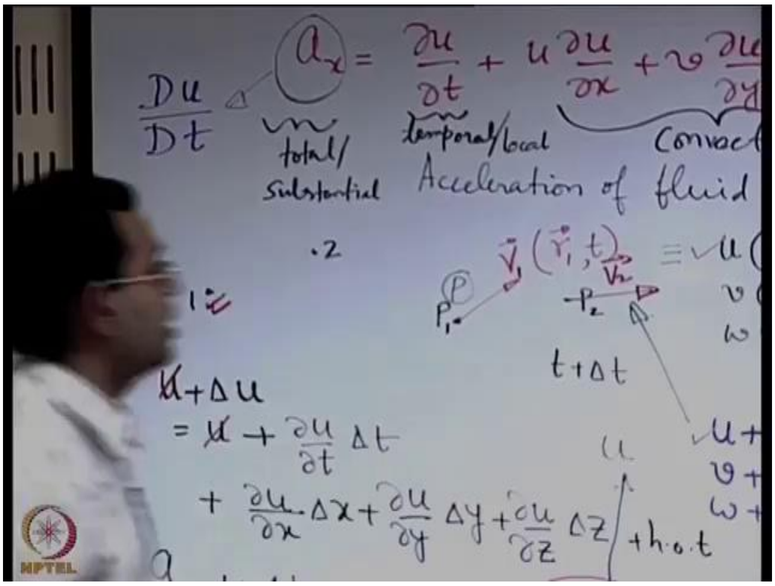

So, $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

Now, if you clearly look into this acceleration expression, there are two different types of terms. One is this type of term which gives the time derivative another gives the spatial derivative. You will see that this expression will give you a first demarcating look of how the expression is different in terms of what we express in a Lagrangian mechanics.

In a Lagrangian mechanics it is just the time derivative that comes into the picture. Here you also have a positional derivative and what do these terms represent; we will give a formal name to these terms. But, before that first let us understand that what these two terms represent. Say you are located at a point 1, now you go to a point 2 in the flow field.

So, when you go there are two ways by which your velocity gets change. How? One is maybe from 1 to 2 when you go you have a change in time and because of a change and you also have a change in position, you have a change in velocity. And, that is solely time dependent phenomena. How can you understand what is the component of the time dependent phenomenon?

If you did not move to 2, but say you confine yourself to 1 say you are not moving with the flow field you are confining yourself with 1, then you are freezing your position. But, still at the point 1 there may be a change in velocity because of change in time, if it is an unsteady flow field. So, because of that it might be having an acceleration.

So, the acceleration that acceleration component is because of what, the time rate of change of velocity at a given point at a given location. So, that is reflected by this one, but by the time when you are making the analysis the fluid particle might have gone to a different point. Even if its local velocity that is velocity at a point is not changing with time, it has gone to a different point there it encounters a different velocity field.

So, here it was encountering a particular velocity field because of its change in position. So, what it has done? It has got advected with the flow, it has moved with the flow and it has come to a new location where it is encountering a different u, v, w. So, because of the change in u, v, w with the change in position it might be having an acceleration. So, that acceleration is not directly because, of the time rate of change of velocity at a given point.

But, because of the spatial change since the particle the fluid particle by the time has traversed to a different location where, it finds a different flow field. And since we are considering that it is an inner tracer particle it has to have the same velocity locally as that is there in a new position.

So, because so the next combination of terms it represents the change in velocity solely due to change in position. So, the total the net change is because of two things: one is if you keep position fixed and you just change time because, of unsteadiness in the flow field there may be an acceleration. The other part is even if the flow field is steady, but you go to a different point because of non-uniformity; because of a change in velocity due to change in position the fluid particle might have a change in velocity. So, the change in velocity in the fluid particle may be because of two reasons. One is because the change in velocity due to change in time even if it were located at the same position as that of the original one.

And the other one is not because of change in time, but because of change in position as it has gone to a different position; because of non-uniformity in the flow field it could encounter a different velocity. And, the resultant acceleration is a combination of these two. So, let us let us take a very simple example to understand it; say you are you are traveling by flight from Calcutta to Bombay. So, when you are taking the flight before taking the flight you see that it is raining very very heavily. And, then say you take 2 2 and half hours you reach Bombay and you find that it is a very sunny weather.

So, the question is now if you if you want to ask yourself a question: does it mean that when you when you departed from Calcutta it was raining in Bombay or when you departed from Calcutta it was sunny at Bombay. Or, when you have reached Bombay is it still raining at Calcutta or is it still is it sunny at Calcutta.

It is not possible to give an answer to any one of these because; the net effect that you have seen is a combination of two things. You have traversed with respect to time so, you have elapsed certain time by which maybe it was raining at Calcutta, but right now it is not raining at Calcutta maybe it was sunny at Calcutta and right now it has started raining. So, it is like at a particular location the weather has changed because, of change in time.

But, the other effect is that you have migrated to a different location and because of the change in location maybe it was before 2 hours raining at Bombay now it is sunny or it might so, happen that it was sunny 2 hours back in moments still it is sunny. So, you can see that individual effects you can maybe try to isolate.

But what is the net combination of changing with respect to position and time; that is the net effect of this and it might not be possible to isolate these effects. So, when you think about the total acceleration so, it is just like a total change. So, when we have the total change it is a change because, of position and because of time and that is why this a_x or maybe a_y or a_z this is called as the total derivative of velocity.

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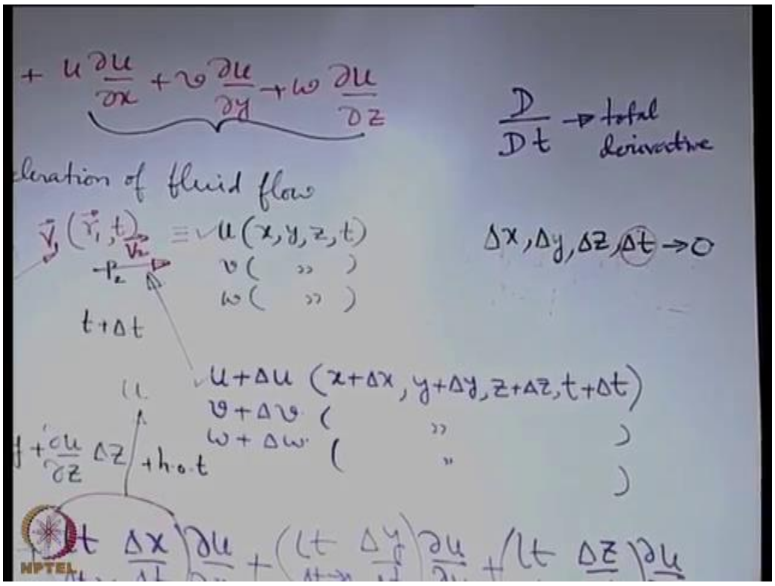

So, $\dfrac{D}{Dt}$ has a special meaning it is called as total derivative. It is to emphasize that it is a resultant change because, of change in position and change in time. So, with respect to change in time if you have a change then it is called as a temporal component of acceleration. Temporal stands for time temporal or transient or local. So, these are certain names which are given; again by the name local it is clear local means confined to a particular position only with respect to change in time and this is known as the convective component.

So, convective component is because of the change in position from one point to the other and this therefore, is the total or sometimes known as substantial. So, the total derivative is a very important concept mathematically here we are trying to understand this concept physically. But, it is not just restricted to the concept of acceleration of fluid flow; it is applicable in any context where you are having an Eulerian type of description.

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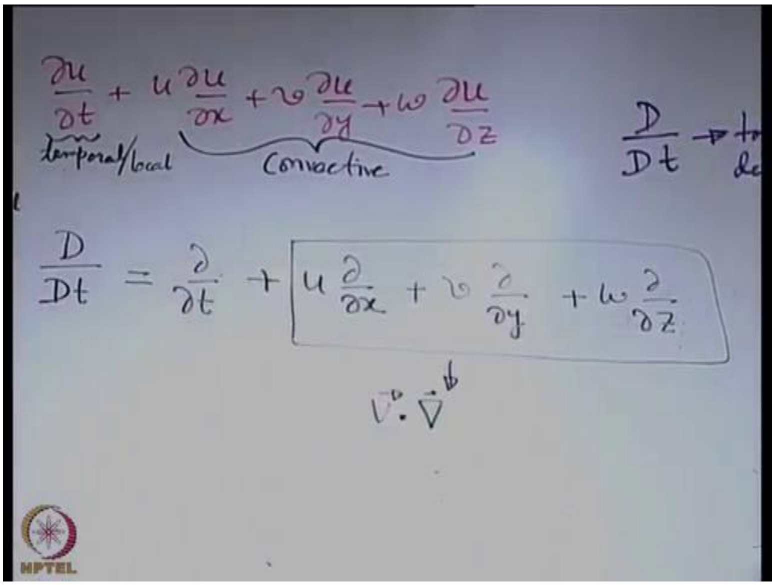

And it is therefore, possible to write the general form of the total derivative as $\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$. So, we can try to answer some interesting and simple questions and see and get a feel of the difference of these with again the Lagrangian mechanics. So, if we ask a question: is it possible that there is an acceleration of flow in a steady flow field, that the flow field is steady, but there is an acceleration.

It is very much possible because, if it is steady only the first term will be 0. But the, but if the velocity components change with position then the remaining terms may not be 0. So, this is a like these are certain contradictions that you will first face when you compare it with Lagrangian mechanics.

In Lagrangian mechanics if there is something which does not change with time, if time derivative is obviously, 0. But, here even if it does not change with time the total derivative it may not be 0. On the other hand it may be possible that it is changing with time at a given location, but acceleration is 0. Because I mean in a very hypothetical case it may so, happen that the local component of acceleration say it is 10m/s^2 convective component is -10m/s^2.

So, the sum of that two is 0, but individually each are not 0. That means, it is possible to have a time dependent velocity field, but 0 acceleration. And, it is possible to have a non-zero acceleration even if you have a time independent velocity field. So, these are certain contrasting observations from the straightforward Lagrangian description.

So, you can write the x component of acceleration in this way and I believe it will be possible for you to write the y and z components which are very straightforward. And, you have to keep in mind that when you write y component this D Dt operator will act on v and when you write the z component it will act on w. So, you can write the individual components of acceleration vector and vector sum will give the resultant acceleration. Now, you can write these terms in a somewhat compact form.

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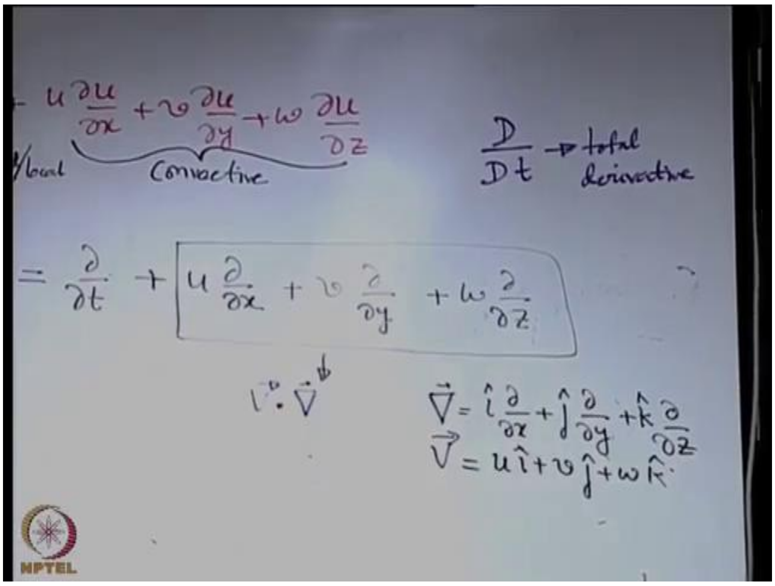

So, this you can also write as $\vec{V}.\vec{\nabla}$ where $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ and V is the velocity vector $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$. So, if you clearly make a dot product of these two you will see that this expression will fall. So, it is a compact vector calculus notation of writing the convective component of the derivative. So, we have got a picture of what is the acceleration of flow, how we describe acceleration of flow in terms of expressions through simple Cartesian notations and maybe also through vector notations.