Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 20
Fluid under rigid body motion

Now that we have seen these stability criteria and so on, we will come to our final topic in Fluid statics which ironically is not fluid statics, but fluid with rigid body motion and as we discussed earlier that we are going to address this issue within the purview of fluid statics, for a very simple reason that when you have fluid under rigid body motion, it is still fluid element without any shear. So, when it is without any shear, it is just the normal force which is acting on the surface. So, the distribution of force in terms of pressure on the surface remains unaltered. No matter whether it is at rest or under rigid body motion; when the fluid elements are deforming, then only you have the shear.

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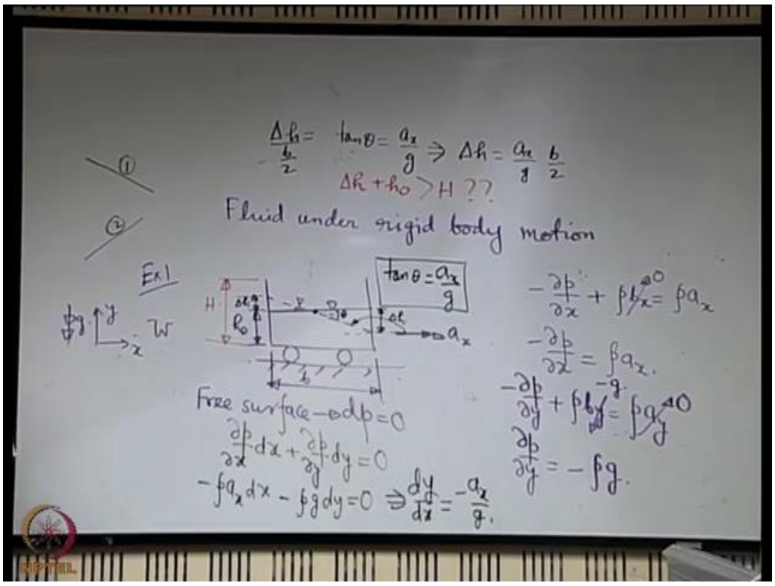

So, let us take a simple example for fluid under rigid body motion to begin with. Let us say that we have a tank, partly filled up with water and the tank is accelerating towards the right with an acceleration of a_x along x fixed acceleration and neglect the deformation of the water that is there in the tank. In reality that deformation is there. So, whatever analysis that we are presenting here is not perfectly correct because in reality there will be shear and deformation and so on. But just as an idealization, just like many times we discussed about frictionless surfaces not that they are there, but through this idealization we learn certain concepts. So, let us say that there is no deformation.

So, the water which is there within the tank just gets deflected in its configuration in terms of the free surface like a rigid body. So, how we calculate that what should be the location new location of the free surface because of this acceleration that is what we want to see. Say initially the height of water in the tank was h 0; maybe let us specify the breadth of the tank as b and maybe the width perpendicular to the plane of the board as w. Let us say that means, it is a rectangular tank. Now, recall that we derived certain expressions with regard to distribution of pressure in presence of body force. $-\frac{d\rho}{dx}+\rho b_x=\rho a_x$

So, $-\frac{d\rho}{dx}=\rho a_x$. Let us say that the y direction. So, we have x and y as our chosen directions. Let us say that the y direction is the direction opposite to the acceleration due to gravity.

$-\frac{d\rho}{dy}+\rho b_y=\rho a_y$ where a_y=0

$\frac{d\rho}{dy}=-\rho g$. The free surface has the same pressure throughout because it is exposed to the atmosphere which has same pressure throughout.

So, there is a pressure equilibrium between that and atmosphere. So, for the free surface dp=0 that should be the governing parameter for locating the free surface. So, when you have dp=0 remember that now p is a function of both x and y. So, you can write $\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=0$

$-\rho a_x dx-\rho g dy=0 \Rightarrow \frac{dy}{dx}=-\frac{a_x}{g}$. $\frac{dy}{dx}$ represents the slope of the free surface.

So, can you tell now whether the slope of the surface will be like this or like this; 1 or 2?

1; because you can clearly see that this will represent a kind of tilt like this. So, this will become the new free surface and the angle that you are considering for the slope is basically this one. Because this is negative a_x is positive and g is positive. So, this is negative. So, it must be an obtuse angle.

So, in the direction in which the tank is accelerating, the liquid will be more down and in the other direction, it will be more up right and if you specify this angle as θ , then $\tan\theta$ is nothing but 180^0 minus this slope angle. So, you can say that $\tan\theta$ is nothing but equal to $\frac{a_x}{g}$ because theta is nothing but 180^0 minus this angle. So, you can find out that what is the extent to which the water level will rise on one side and maybe fall on the other side? Let us say that this rise is Δh on one side because of symmetry, it will be a fall of equivalent Δh on the other side. So, it will be as if swivelling with respect to the centre. So, we can calculate what is Δh.

$$\frac{\Delta h}{\frac{b}{2}} = \tan\theta = \frac{a_x}{g} \Rightarrow \Delta h = \frac{a_x}{g}\frac{h}{2}$$

Let us say that the total height of the tank is h; if it so happens that let us say for from calculation we get $\Delta h+h_0$, it is greater than h. Practically that is not possible right because the liquid cannot occupy a height which is which is greater than that is provided by the tank. So, what this will mean? This will mean some water has spilled out.

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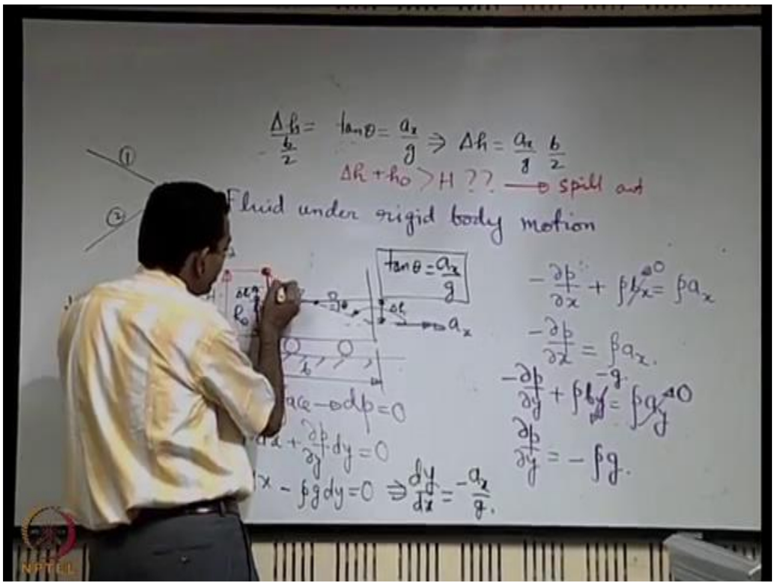

So, this is a condition from which you can say that it has actually spilled out. When it has spilled out, it is no more this configuration. So, when it has spilled out; what will happen, what type of configuration you expect? So, before spilling out it tried its best to climb up to the top most level and then it spilled out.

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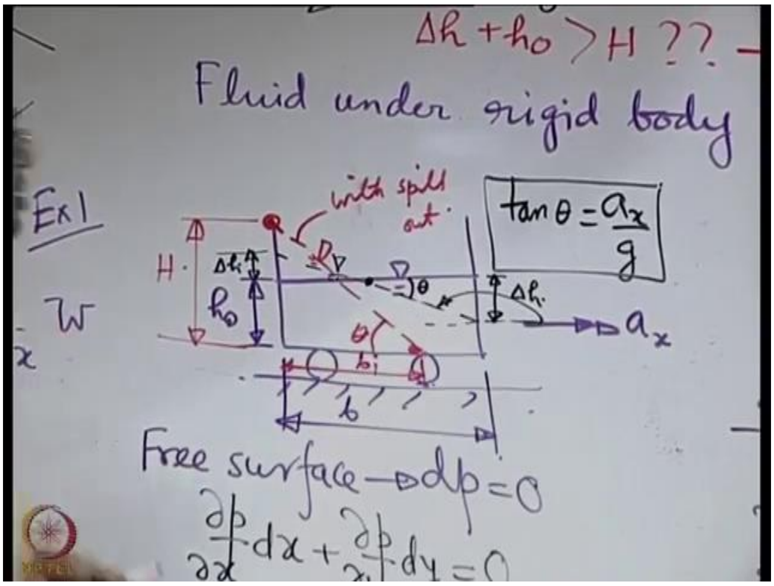

So, this will be one end of the surface and maybe the other end is like this somewhere. Irrespective of whether it has spilled out or not, you still have this expression applicable. So, now, this will be the angle θ . You can find out that what should be this length? Let us say b_1. $\frac{H}{b_1} = \tan\theta = \frac{a_x}{g}$. H being the height of the tank known, so from there you can find out what is b_1.

And therefore, you can calculate what is the volume which is there now within the tank and the difference between the original volume and that volume will give you what is the volume that has got spilled. So, you see that it is not just like your solution should not be driven by a magic formula, but based on the numerical data given, you have to come to a decision whether the water is there inside or it has got spilled and so on. Let us take a variant of this example.

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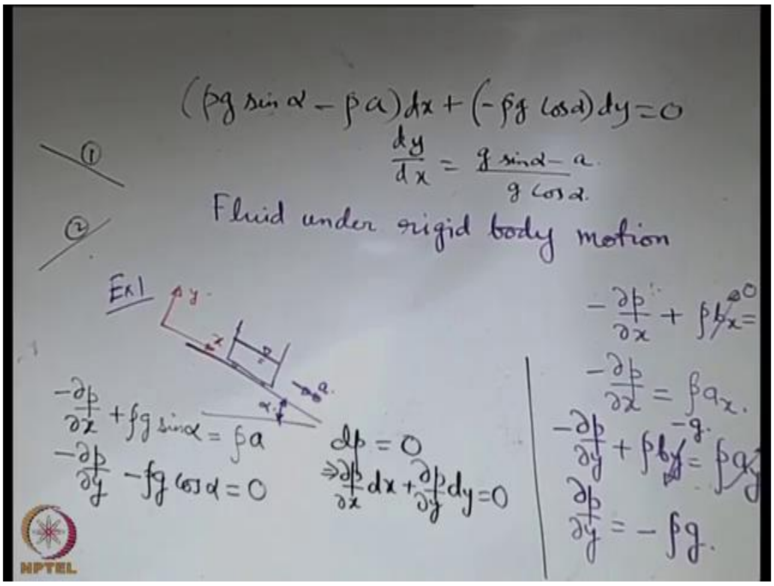

So, the variant of the example is that now we have the tank located on an inclined plane with an angle of inclination α, the tank is there on the plane and it is accelerating say downwards with an acceleration of a, which is a uniform acceleration maybe because of a resultant force which acts along that direction.

So, in this case it may be more convenient, if you fix up your coordinate axis relative to the inclined plane say x and y. So, the similar equations will be applied and let us just do it very quickly because it is very straightforward.

$$-\frac{\partial p}{\partial x}+\rho g\sin\alpha=\rho a \qquad dp=0$$
$$-\frac{\partial p}{\partial y}-\rho g\cos\alpha=0 \qquad \frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=0$$

$$(pg\sin\alpha-\rho\alpha)dx+(-\rho g\cos\alpha)dy=0$$
$$\frac{dy}{dx}=\frac{g\sin\alpha-a}{g\cos\alpha}=\tan\theta$$

where θ is the angle relative to the original location of the free surface or like the assumed x direction. So, you can see now that there is depending on the magnitude of a, this may be positive or negative right. So, you may have a case when $g\sin\alpha$ is greater than a or $g\sin\alpha$ less than a.

See here what we are doing? Here, we are implicitly applying the same condition. It is you see that we are having one very important assumption. The assumption is that you donot have an oscillation in the surface. So, sometimes because of this displacement of surface oscillates, it becomes like a wavy situation and that is known as sloshing of tanks.

So, we are not going into that details. So, we are assuming that the surface remains flat and when the surface remains flat and under these conditions when you have dp=0 that is exactly the same condition.

So, it is better to be habituated with these. Because again, I am telling you there are situations when it will not be as straightforward as these. So, you have to use these fundamental equations. So, whenever you are solving a problem try to adhere to the fundamental situations. I am getting your point; why you are telling this because you have been habituated in solving problems in that way through your entrance exams. But we will be encountering more challenging problems than what you have solved earlier through that type of magic situation and we try to avoid that. So, our basic intention will be that we have this basic equation. This should slowly guide us for solving whatever complicated problems that we are having of this type. Let us consider another one.

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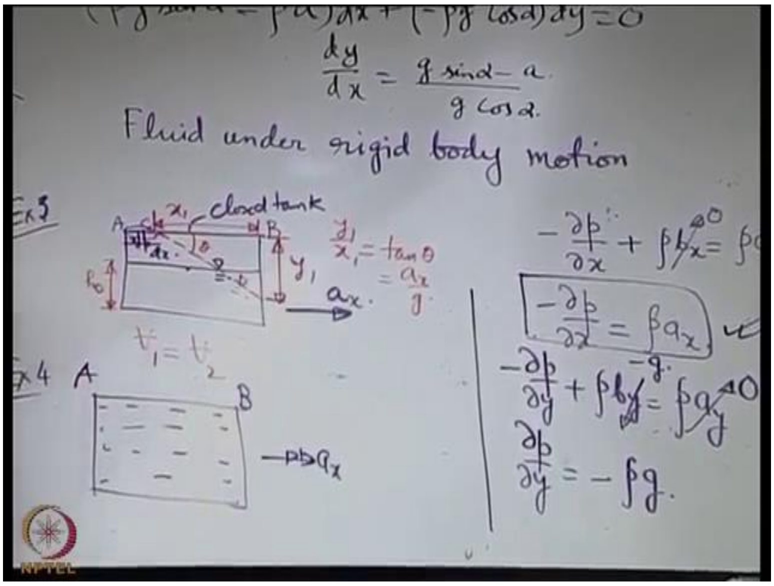

Say example 3. Now, let us say you have a close tank ok; close tank partially filled with water. This is closed. Again, you are doing the same thing accelerating it along x. What will happen to the free surface? Now, there cannot be any spilling.

So, when there cannot be any spilling, there is even a chance that the free surface is like this. So, the symmetry with respect to the central line which was there for the previous case without spilling is now broken, that symmetry is not guaranteed because it may try to escape and since it is not finding an escape route, it will break the symmetry and get distributed in what way? In such a way, that the volume of the liquid now is conserved because it cannot go out of the tank.

So, if you say that this dimension is say y_1 and this is x_1, $\frac{y_1}{x_1} = \tan\theta = \frac{a_x}{g}$ and you can calculate y_1 and x_1 by considering that the volume which was there originally is same as the volume of water after this tilting of the interface. So, this is the new interface.

Now, let us say that we are interested to find out what is the total force acting on the top surface or the lead of the container. How will you find it out? So, you let us say this point is C. So, you have to keep in mind that only up to AC, it is in contact with the fluid. So, you can find out the pressure distribution as a function of x from A to C. Take a small element at a distance x from A of width dx.

Let us consider maybe a fourth variant which I will not solve, but just tell you that such a variant is also possible. Say you have a tank now it is completely filled with water and it is closed.

You can you assume any one of the point say A as the reference pressure say p some reference p because always when you have pressure its relative to some point and so you can express the pressure distribution in terms of the pressure at the point A and then, integrate it over the element. So, next example is you have this tank accelerating towards the right, but completely filled with water ok, but closed. So, water cannot escape.

Here, you do not have the botheration of finding out what is the portion that is exposed with the liquid because since it is completely filled, it should be completely exposed. Now when we consider the rigid body motion, it is not always just translation; it may also be rotation. So, let us consider a rigid body rotation example.

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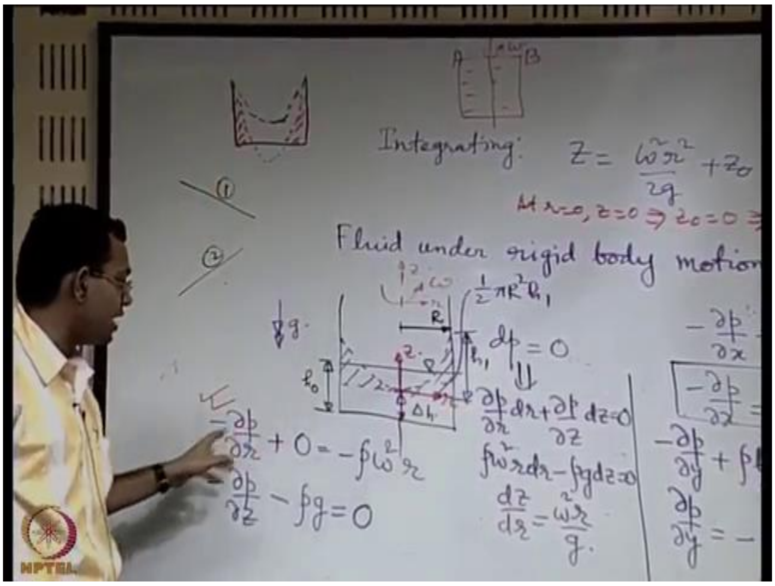

This type of example you have seen earlier that you have a tank filled up with water to some height h_0, it is a cylindrical tank of radius R.

So, we are using a r z coordinate system, axis symmetric coordinate system say this is r coordinate and along this there is z coordinate. So, this tank is rotated with respect to its axis with an angular velocity, ω. So, when it is rotated, you already know it that it will come to its free surface we will come to a deformed or deflected shape like this which is a paraboloid of revolution. Let us quickly see that how we can derive that it should be a paraboloid of revolution from these fundamental principles, not from any magic formula.

So, there are two directions r and z.

$$-\frac{\partial p}{\partial r}+0=-\rho\omega^2 r$$

Say you are standing on a platform and you are looking into it from a inertial reference frame. This is not an external force that is applied that you can see only when you have a rotating reference frame that is attached to the platform that is having angular motion.

So obviously, there will be no br that you have to keep in mind. This is the form of the Newton's Second Law in an inertial reference frame. With a non-inertial reference frame, you have to use a pseudo force for the inertia force; but that then you do not write equation of dynamics, but equivalent to equation of static equilibrium, you convert that into an equivalent static equation through D'Alembert principle, but here we are not talking about that. We are talking about the proper acceleration. So, you have no body force, but you have acceleration.

See if you had considered it in a rotating reference frame, the right hand side would be 0 because you are writing static equilibrium; this would be represented by the pseudo force. So, the final equation would eventually be the same. It is just a matter of the reference frame with respect to which you are writing, but this is written with respect to the inertial reference frame.

$$-\frac{\partial p}{\partial z}-\rho g=0$$

Again like whenever it is vertically moving and so on you can substitute.

So, start with this basic equation depending on whatever information is given in the problem you try to use it. So, for the free surface you have $dp=0 \Rightarrow \frac{\partial p}{\partial r}dr+\frac{\partial p}{\partial z}dz=0$.

$$\rho\omega^2 r dr-\rho g dz=0$$
$$\frac{dz}{dr}=\frac{\omega^2 r}{g}$$

You can now integrate this with respect to r. So, on integration what follows let us just complete that. $z=\frac{\omega^2 r^2}{2g}+z_0$

So, if we choose this as our origin of the coordinate. So, this is r and this is z. So, if you have at r=0, z=0 $\Rightarrow z_0=0 \Rightarrow z=\frac{w^2 r^2}{2g}$. This formula you have encountered earlier. So, we can clearly see that it is a its an equation like of a parabola. So, it is a parabola of revolution is a 3 dimensional situation and you can calculate the other things just similar to what we did in the previous case with an understanding of what that again if it does not spill whatever was the volume that should be conserved.

So, how can you calculate the volume? So, initial volume you can calculate; initial volume is $\pi R^2 h_0$.

Final volume is whatever is the depression say Δ h plus the volume of the shaded parabola of revolution. If this height is h 1, then the shaded volume will be $\frac{1}{2}\pi R^2 h_1$. By simple integration, you can find out this volume. So, by equating the initial and the final volume, you can find out totally the deflected configuration.

Again, you may have to check that $h_1 + \Delta h$, if it becomes greater than the height of the original tank; then, it will spill. And spilling again may have two different cases. You may have in one case the cylinder is rotating in such a way that you have spilling, but still it is having an inter phase shape like this. Again it may so happen that it is rotating so fast that it is spilling, but only the part of the paraboloid of revolution is within the cylinder. So, then it is an imaginary paraboloid of revolution even outside that you have to consider to find out what is the volume that is there inside. So, it all depends on the rotational speed C to the given dimensions and so on.

So, it is not just like a fixed formula, but you know what is the basic principle we have discussed enough numbers of examples to see that what is the basic principle and that basic principle should guide you to find out that what is the case. Now if it is totally closed cylinder as a final example say we have a cylinder that is totally closed and filled up with liquid and rotated with respect to its axis. What is the total force on the top lid A B? Again, the basic principle is the same you just use this $\frac{dp}{dr}$ ormula to find out how pressure varies with r. Of course, with a reference say at $r=r_0$; $p=p_0$ with a reference because pressure you always calculate with a reference.

So, you know how p varies with r by integrating this with respect to r; when you integrate with respect to r , z is fixed. So, then you can find out the total force by taking an element. Here an element will be $2\pi r dr$.

So, integrating over that you can find out the total force on the top surface. Total force on the bottom surface will be that plus the weight of the fluid which is there right.

So, we will close this discussion by seeing just one maybe one example, where we will see that how thos type of vortex motion is generated in practice.

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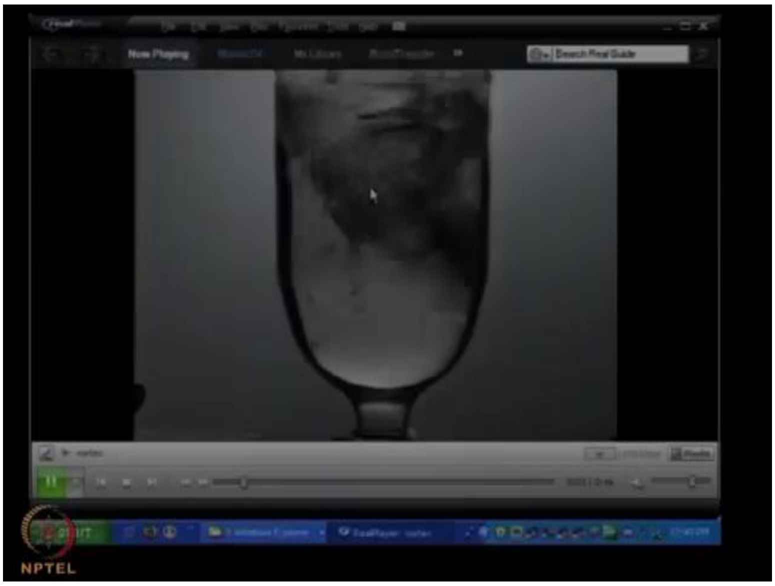

You see that it is you see the this type of a vortex motion that is there you I mean not exactly a paraboloid of revolution, but it is by rotating the fluid in a container; why it is not exactly a paraboloid of revolution is because we have neglected here the viscous effects. The shear between various fluid layers, we have assumed that the fluid rotates like a rigid body. In reality that is not the case and we will look into these situations more emphatically whenever we are discussing with viscous flows.

So, we close our discussion on fluid statics with this.