Introduction to Fluid Mechanics
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Lecture – 14
Governing equation of fluid statistics

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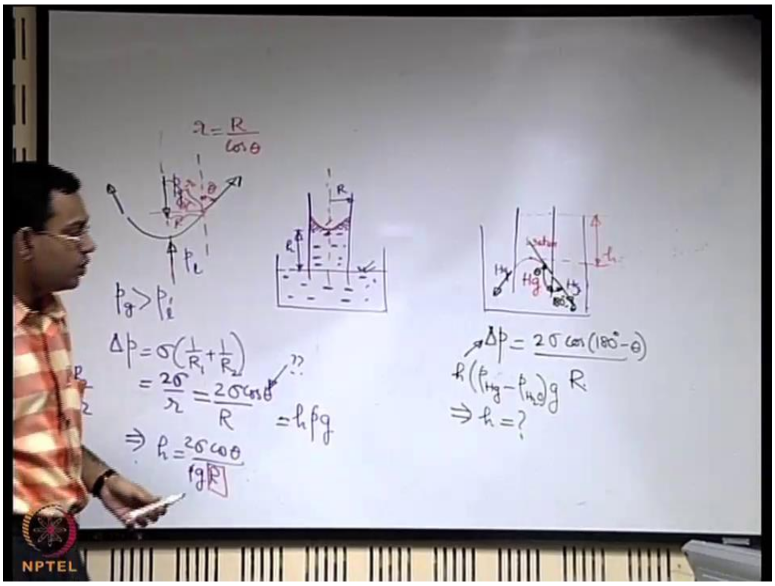

With this background, now we will move into more general considerations for equilibrium of fluid elements. Here whenever we were discussing the surface tension force, we were assuming that the pressure is being distributed in a particular way. And we were intuitively using some concept of high school physics that if you have a depth of h, then what should be the variation in pressure because of that depth of h.

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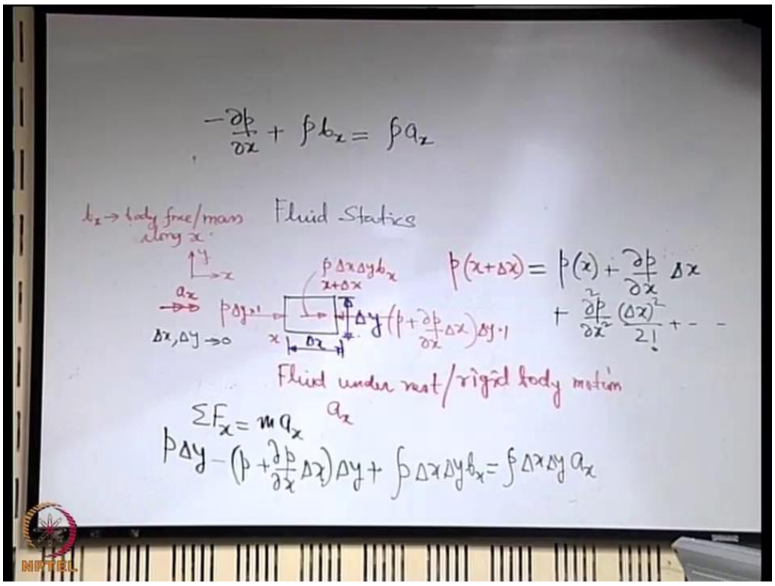

Now, we will look into it more formally. So, we will go into the understanding of fluid statics. We start with an example of a fluid element, which is in static equilibrium. And we take an example of a two-dimensional fluid element just for simplicity. We have taken such examples earlier what these examples signify that you have a uniform width in the other direction perpendicular to the plane of the board.

So, let us say that $\Delta x, \Delta y$ are the dimensions of this fluid element, remember that this fluid element is at rest. When this fluid element is at rest that means, we are sure that it is non-deforming, because deforming fluid element is definitely not a fluid element at rest. And when it is non-deforming, we are clear that there is no shear which is acting on it that means, there is only normal force, which is acting on all the phases of the fluid element.

So, we can designate the state of stress on each phase of the fluid element by pressure. So, we are only writing forces along the x direction, similar equations will be valid for the y direction. When you have say the left phase under consideration, just like this; let us say that p is the pressure on the left phase, and the force corresponding to that is $p\Delta y$ into say 1, which is the width of the fluid element.

When you come to the opposite phase, we are bothered about these phases right now, because we are only identifying the forces acting along x, because we will write equation of equilibrium along x not that forces are not there on the other phases. So, this is not a complete free body diagram, it only just shows the forces along x direction. We are not really committed to what are the other forces, which are acting on it, there are many other body force which is acting on it along x and y.

Mathematically, speaking what question we are trying to answer, we have a function here say p, we want to find out the value of the function at a different location, say this location is x.

We are interested to now find out the value of the function at $x+\Delta x$ in terms of what is the value of function at x, the function here is p that means we want to see that what is p at $x+\Delta x$ in terms of what is p at x. And that we can easily do by using a Taylor series function, so that we will do we will write $p(x+\Delta x)=p(x)+\frac{\partial p}{\partial x}\Delta x+\frac{\partial^2 p}{\partial x^2}\frac{(\Delta x)^2}{2!}+...$ There are infinite number of terms, but as you take Δ x very small, maybe you may neglect the higher order terms in comparison to the dominating term and the gradient.

Keeping that in mind that we are treating with cases, where $\Delta x, \Delta y$ are very small; so, $\Delta x, \Delta y$ all tending to 0. So, this will become from the expression that we have here what we can write these will be p that will be the pressure here, we will keep these in mind. So, latter on whenever we encounter any function, we will use the Taylor series expansion to identify what is the change that is taking place across different phases of fluid elements, because that we will have to do very commonly in many of our analysis. So, this multiplied by the area on which it is acting is the force due to pressure on this phase.

Let us say that there is a body force, which is also acting on the fluid element. So, the body force let us say that b_x is the body force per unit mass acting along x. So, what will be the total body force, which is acting on this along x first you have to find out what is the mass of fluid element what is that, it is the density times, the volume of the element $\Delta x\Delta y$, so this is the mass of the fluid element that times the body force per unit mass gives the total body force along x. So, these are the forces, which are acting on fluid element.

Now, let us try to answer another question are these still the force only forces, which are acting if the fluid element is under rigid body motion that is the fluid element is moving like a rigid body, there is no internal deformation, but as a whole it is just like a solid that is getting displaced that may be displaced linearly or angularly, but it is having a motion, but the motion is a rigid body motion.

The difference between a fluid element at rest and fluid element under rigid body motion is that when it is under rigid body motion, might be having like a velocity, acceleration, and so on.

But, in terms of the surface motion which are acting, if the fluid element is non-deforming, then there is no shear component of force. So, for a non-deforming fluid element, there is no difference between the surface forces, which are presented in this diagram, and the surface forces which are there when it is say moving with an acceleration. So, this type of forcing description is equally valid, if the fluid is under a rigid body motion.

So, we identify this situation not just at fluid under rest, but also rigid body motion. We will see such examples, where the rigid body motion of the fluid will be very interesting like you may have rotation of a fluid element like a rigid body, and we will see that what kind of situation it creates.

So, broadly this is also studied under the category of fluid statics not because it is a static condition, but in terms of the characteristic of the fluid the deformable nature is not highlighted here. And that is why; we may use broadly similar concepts. And, we will learn these concepts together under the same umbrella, because they are very very related. In one case it will, it may have an acceleration, in other cases it may not be, otherwise it is very very similar.

So, let us say that it is under rigid body motion, and therefore let us it has still some acceleration along x, say ax is the acceleration which is there along x. So, we can write the Newton's second law of motion for the fluid element, and $\sum F_x = ma_x$

So, $p\Delta y - (p + \frac{\partial \rho}{\partial x}\Delta x)\Delta y + \rho\Delta x\Delta y b_x = \rho\Delta x\Delta y a_x$. $\Delta x\Delta y$ will get cancelled from both sides these are small, but not equal to 0 these are tending to 0.

So, $-\frac{\partial p}{\partial x} + \rho b_x = \rho a_x$ is the expression which relates the pressure gradient with the body force that is acting, and if there is any acceleration that acts on the fluid element it is having similar expressions are valid for the motion along y. So, we are not repeating it again. With this kind of a general idea, so this is a very general expression. This general expression just considers that there is a body force, on the fluid is having some acceleration in a particular direction subjected to the body force, but it is a non-deformable fluid element.

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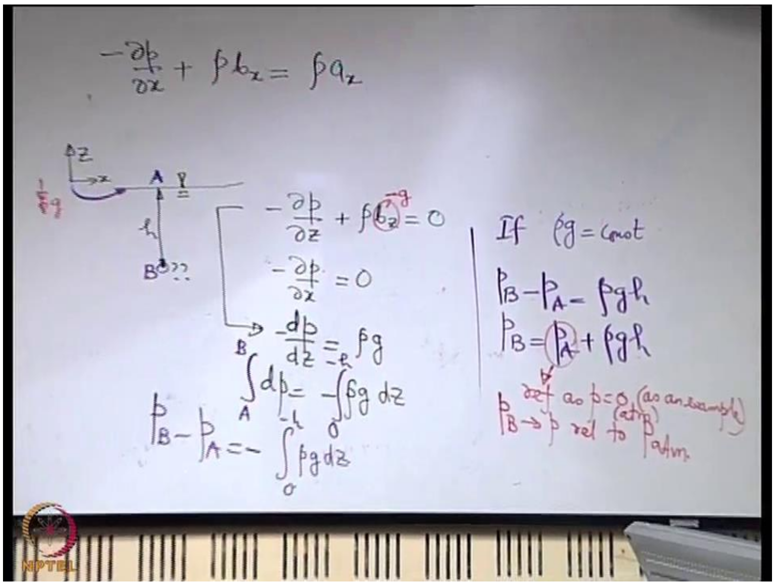

With that understanding, we will try to identify that what is the variation of pressure just due to the effect of gravity as a body force in a fluid element at rest. We consider that there is a free surface of a fluid, $\underline{\underline{\nabla}}$ is a symbol in fluid mechanics that we will be using to designate a free surface, this is a triangle with two horizontal lines very short horizontal dashes or lines at the bottom. This a kind of a technical representation of a free surface.

We consider that we are interested about some depth usually that direction in which depth varies is typically taken as z direction, this is just a common notation in most of the books that the vertical direction across which the gravity is acting. Of course, the opposite to the action of gravity, because gravity will be vertically downwards and opposite to that is considered as a z axis, just as a common notation.

So, let us try to write this kind of a equation for this fluid which is at rest, it is of a substantial depth. So, we are interested to find out what is the pressure at this point, which is at a depth h from the free surface. There is no acceleration of this fluid, it is under absolute rest. So, the a_x or here a_z term will be 0, so $-\frac{\partial p}{\partial z} + \rho b_z = 0$. When you have the z direction, you also have a horizontal direction like x, and for x you can write again similar expression.

So, $-\frac{\partial p}{\partial x}$ = 0. There is no body force which is acting along x, because only body force to which this is subjected is the gravity. So, there is no body force along x, and there is no acceleration that is it is having along x. So, the second expression is even more simple, but it gives us a very important insight what is that that if you are not having a body force along a particular direction and the fluid is under rest, then pressure does not vary within the same fluid along that direction that means, for a horizontal along a horizontal line, you are not having any pressure variation in a continuous fluid system.

And this is one of the basic principles that we use for measurement of differentials as you have seen in examples of manometers earlier. So, this is something which is of great consequence, but it is obvious conclusion.

$$-\frac{\partial p}{\partial z} = \rho g \Rightarrow dp = -\rho g dz$$

So, if we want to find out, what is the pressure difference between set two points A and B. So, we have to integrate $\int_A^B dp = -\int \rho g dz \Rightarrow p_B - p_A = -\int_0^h \rho g dz$

Now, it is important to see that: what is the length scale that we are considering over which this variation is taking place. If this h is quite large, there may be a significant variation in density over it, just like consider the atmosphere which is above the surface of the earth as you go more and more above the surface of the earth you expect the density to change, because the temperature changes and so on.

And therefore, the density in many cases may not be treated as a treated as a constant. So, if it is treated as a constant, then it can only come out of the integral. Similarly, you also are probably working on length scale over which g is not changing. If you are taking a large height, like if what people who are dealing with the atmospheric sciences for them, the length scales are large length scales over which you may have even a change in acceleration due to gravity.

If $\rho g = const, \quad p_B - p_A = \rho g h \Rightarrow p_B = p_A + \rho g h$

Now, important thing is that see we are not expressing the pressure at B just in absolute sense, we are expressing it in relative to the pressure at A. Many times this pressure at A sets at this is atmosphere, so that may be taken as a reference. So, if this is taken as a reference, as p equal to 0 as an example. So, whatever is the atmospheric pressure, say we call it 0 that means, any other pressure we are expressing relative to the atmospheric pressure.

So, then in that case P_B is the pressure relative to p atmosphere. If p equal to 0 is the atmosphere, it is not definitely equal to 0. So, any pressure which is expressed in terms of the atmospheric pressure that is a relative way of expressing the pressure, it is not that always you have to express relative atmospheric pressure, but atmospheric pressure being a well known standard under a given temperature. So, reference with respect to atmospheric pressure is something which is a very standard reference that we many times use.

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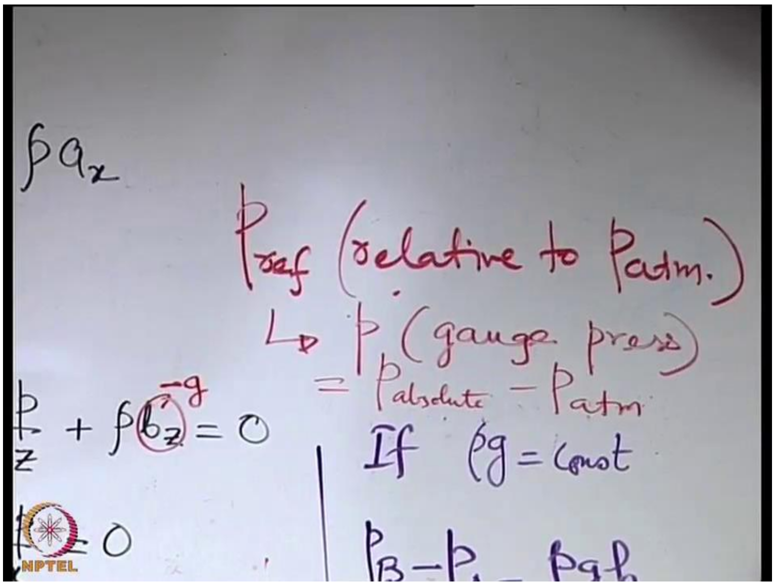

So, reference pressure relative to atmosphere, so we are talking about a reference, where the reference is the atmosphere. Then whatever pressure is there at any other point, we call that as a gauge pressure. So, this is just a terminology. So, gauge pressure means that any pressure relative to atmospheric pressure, $P_{gauge} = P_{absolute} - P_{atm}$. There is a difference between the absolute pressure and the atmospheric pressure that is as good as taking the atmospheric pressure as zero reference, and mentioning the pressure relative to that.

So, obviously ρg is a constant. And there is no other body force, which is acting on it and fluid is at rest that is these are the assumptions that are there with such a simple expression. Now, with this kind of a concept one may utilize device one may utilize this type of concept in making devices for measuring atmospheric pressures just like you have barometers.

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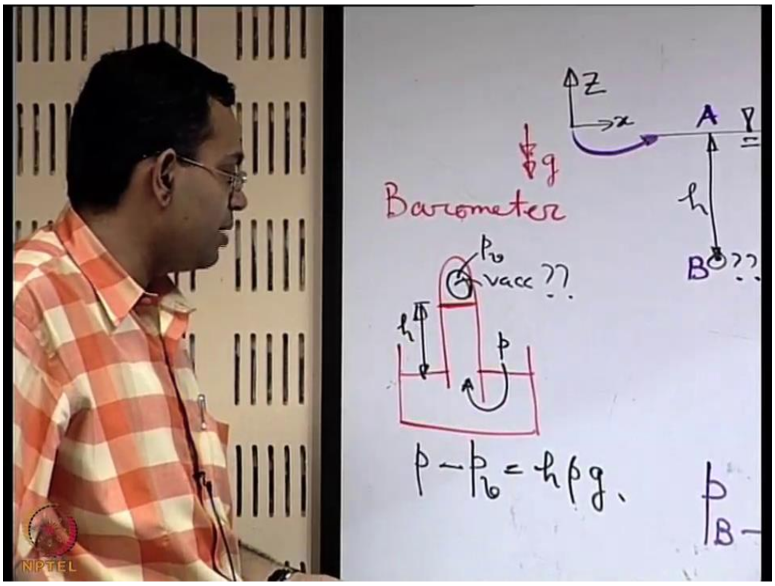

Whenever we will be learning a concept, we will try to give examples of measurement devices, which try to utilize those concepts. As all of you know barometer may be utilized to measure the atmospheric pressure. So, how it is there, you have say inverted tube, and this inverted tube is say put in a bath of some fluid say mercury, and let us say that it is there up to this much height.

Now, this much of height is there are various forces, which are acting on this. So, one is you have we are of course neglecting the local surface tension effect, and the capillary formation. You must keep in mind that as this radius becomes smaller and smaller, the effect of the curvature may be more and more important, because surface tension effect will be more and more important, and there are many significant errors in reading because of that.

Now, if we just neglect that effect for the time being, then you have atmospheric pressure acting from this side. If we assume that there is a vacuum here, there is a big question mark, whether there will be vacuum or not, but let us for the sake of simplicity assume that there is a vacuum. Then whatever pressure is there, which is acting from this side that is balanced by the height of the liquid column, which is there on the top. So, from that you can get an estimation for what is the pressure here, let us say that p is the atmospheric pressure. So, $P \times surface$ area on which it is acting is the force that is being sustained by the weight of the liquid column.

So, it is like $h\rho g$ that into the area, the area gets cancelled from both sides. So, you get this p, if it is vacuum as the $h\rho g$. But, if it is not a vacuum, let us say that there is some pressure here, which is the vapor pressure of the fluid which is occupying this and it is common that such vapor pressure will be there. Because, if it is a saturated liquid, it is likely to have its own vapor on the top of that and that will always exert some pressure. So, it is never a vacuum in an ideal sense.

So, $P - P_v = h\rho g$. So, if there is a vapor pressure, you cannot just use h rho g for the estimation of the atmospheric pressure, but we have to make a correction because of the presence of the vapor and that is the function of the temperature, because vapor pressure varies with temperature.

Very commonly the mercury is one of the fluids that is being used for this purpose. And why mercury is being used, obviously because it is quite dense, it will not occupy a very large height for representing the atmospheric pressure. If you use any other fluid, it may occupy a great height, so it may be an unmanageable device unmanageable long device. Also the vapor pressure of mercury is quite small in most of the temperature ranges.

And therefore, this correction is not that severe. These two are the important reasons, there are many other reasons which are always into the picture, when you select a fluid for measurement of a pressure, so it like in a barometer.

Now, we have found out a particular way in which you have a estimation of variation pressure because of body force which is acting, and for fluid elements which may be at rest or subjected to acceleration. So, we will utilize this principle to calculate two important things. One is if there is a plane surface, which is immersed in a fluid. What is the total force, which is there on the plane surface because of the pressure distribution.

Now, we have realized that pressure is like a distributed force, because it varies with the depth. So, there are different depths, you have different pressures. Therefore, it will be like a simple statics problem, where you have distributed force on surface. To find out what is the total force, which is acting. If you have a curve surface, we will see that the technique may be different, but broadly we can utilize some of the concepts of pressure distribution on a plane surface even to calculate force on curve surface. So, in the next class, we will see what are the forces on the plane and curve surfaces which are there in a fluid at rest. And then to see that what are the consequences, and we will work out some problems related to that. So, we stop here today.

Thank you.