Robotics
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Lecture - 25
Robot Dynamics (Contd.)

Now I am going to discuss how to determine the inertia tensor for the robotic link having circular cross section.

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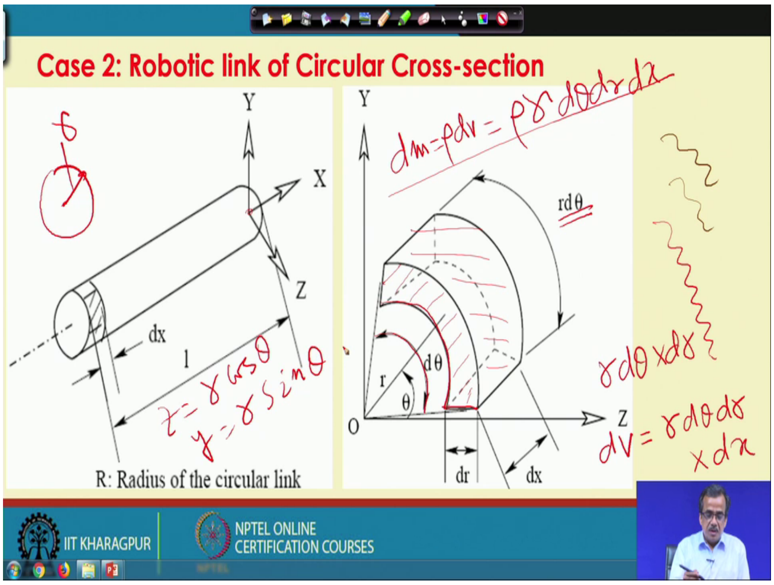

Their length of the robotic link is s into equal to l and it is having the circular cross section with the radius r. Now here I am just going to consider a small element, the same element I am just going to redraw here. So, this is a X Y and Z. So, the coordinate system is attached here.

Now, let us try to concentrate on this small element, now this is r this particular included angle is your b theta. So, this arc is nothing but your r d theta. So, this is nothing but is your r d theta and this is a dr. So, the cross sectional area of this part is shaded part is nothing but is your r d theta multiplied by dr and so, to determine the volume that is your dv. So, what we do is; so this area r d theta dr multiplied by dx. So, this is nothing but the volume.

So, if I want to find out the differential mass that is dm is nothing but rho dv and that is nothing but is your rho r then comes your d theta dr dx. So, this is nothing but the differential mass of this small element. So, the differential mass of this small element is nothing but rho r d theta dr dx. And now let us try to find out its moment of inertia and before that let me write down, that this particular z, Z is nothing but. So, this is a r this is theta. So, Z is r cos theta and your Y is nothing but r sin theta.

Now, using this particular expression, now we can find out the moment of inertia.

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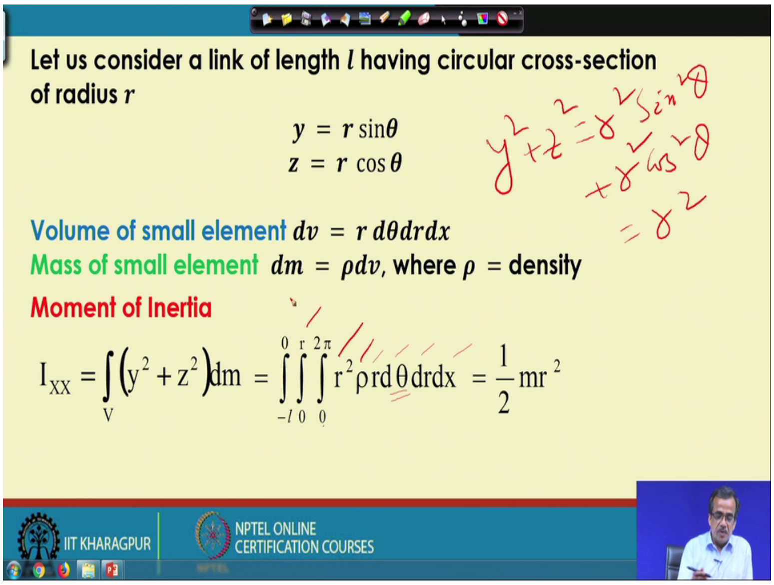

Now this your differential mass as I told a rho dv that is a rho r d theta dr dx where rho is the density. Now moment of inertia about xx that is a I xx is nothing but volume integration Y square plus Z square dm. Now Y square plus Z square is nothing but your Y square plus Z square is nothing but Y square plus Z square is nothing but r square sin square theta plus r square cos square theta. So, this is nothing but your r square. So, this is nothing but r square.

So, Y square plus Z square is nothing but r square and this particular dm is nothing but rho then comes your r d theta dr dx and let us try to find out the limit of this integration now d theta. So, theta will vary from 0 to 25, then comes your r r will vary from 0 to r and X will vary from minus l to 0 now let us let us try to see what happens like how to decide the range for this particular x.

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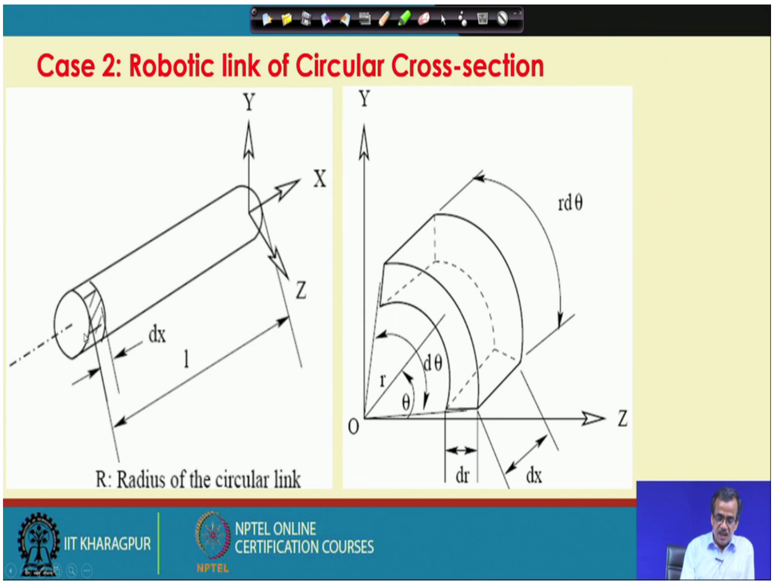

Now, here we have got the origin of the coordinate system.

So, it is from minus l to 0 that is the range for your X now if this is the range for X.

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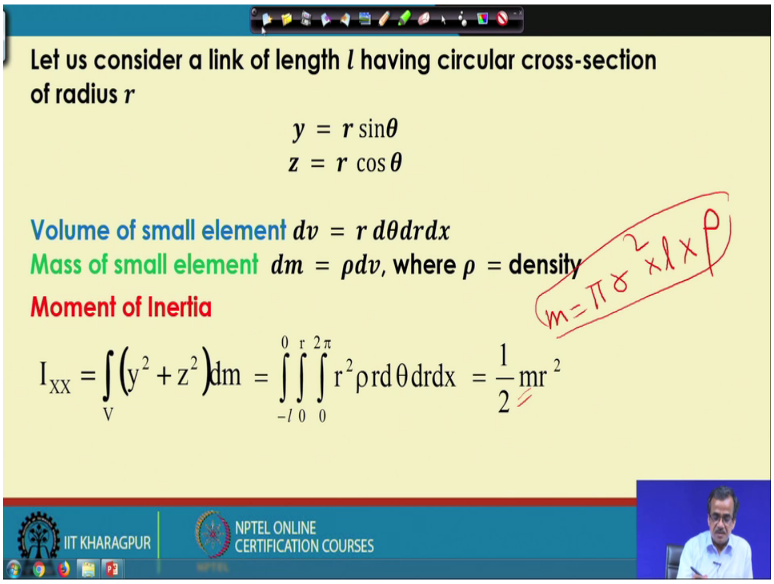

So, we can find out this and we can carry out this integration. And if we carry out this integration we will be getting half m r square, where m is nothing but the mass of this particular the link having a circular cross section. Now this mass m can be determined as pi r square multiplied by l is the volume multiplied by rho that is a density is nothing but the mass of this particular the link.

So, half m r square is nothing but the moment of inertia about xx.

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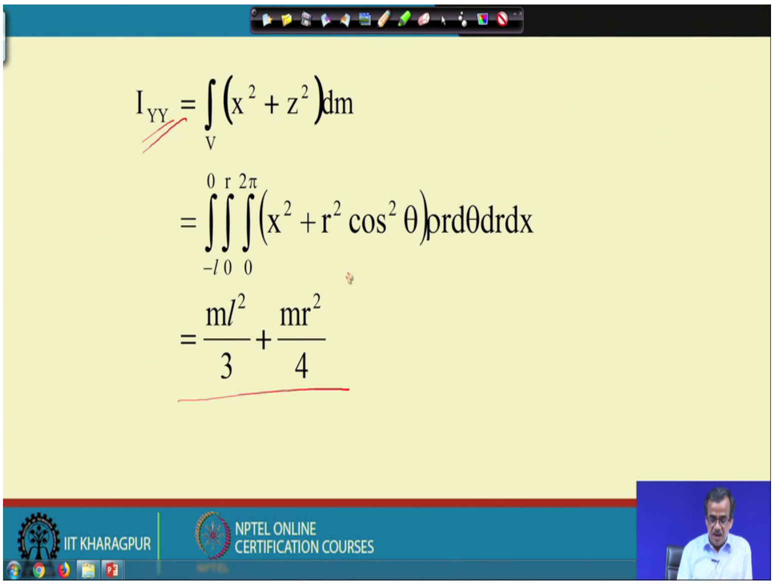

Now if I see this moment of inertia about yy. So, that is nothing but the volume integration Y square plus Z square dm. So, sorry s square plus Z square dm. So, in place of Z square I am putting r square cos square theta, and next is your dm is rho r d theta dr dx and if you carry out this particular integration and if these are the limits for integration. So, very easily you can find out. So, this particular expression for I yy that is nothing but the moment of inertia about yy that is a ml square by 3 plus mr square by 4.

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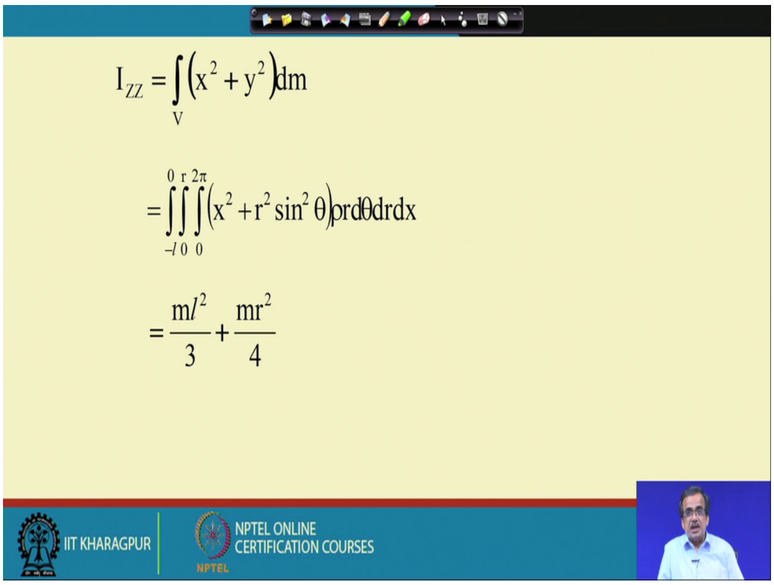

So, following this I can also find out the moment of inertia about zz and that is nothing but volume integration X square plus Y square dm and a Y square is nothing but r square sin square the square theta then rho r d theta dr dx. And if we carry out this particular integration we will be getting ml square by 3 plus mr square by 4. So, I can find out the moment of inertia about zz the next is the product of inertia.

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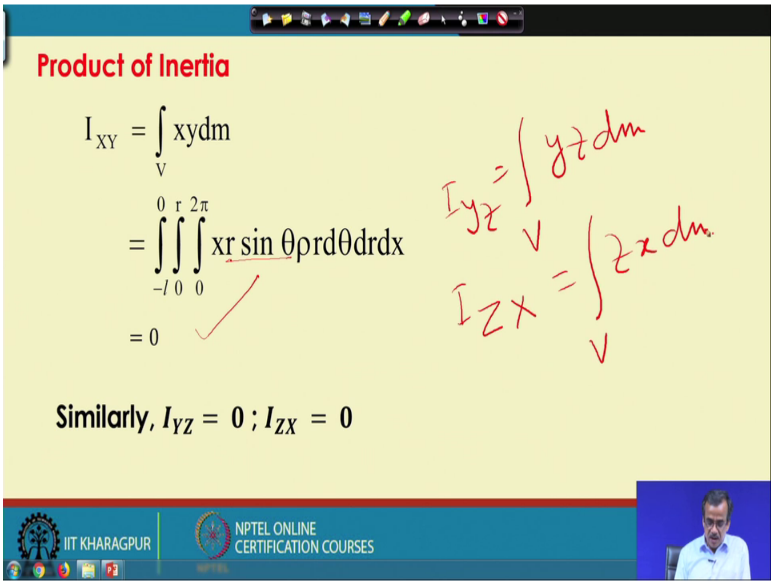

The product of inertia I xy is nothing but the volume integration xy dm.

Now here, Y is nothing but your r sin theta and dm is rho or d theta dr dx and these are the ranges for the integration and if you carry out this integration, we will be getting I xy is equal to 0. Now by following the similar method we can also find out what is I yz. The and I yz is nothing but is your the volume integration yz dm and if you carry out this integration I will be getting I yz is equals to 0. Similarly I can also find out the product of inertia that is I zx and that is nothing but your Zx dm. And if I carry out this particular integration and we will be able to find out that I zx is equal to 0.

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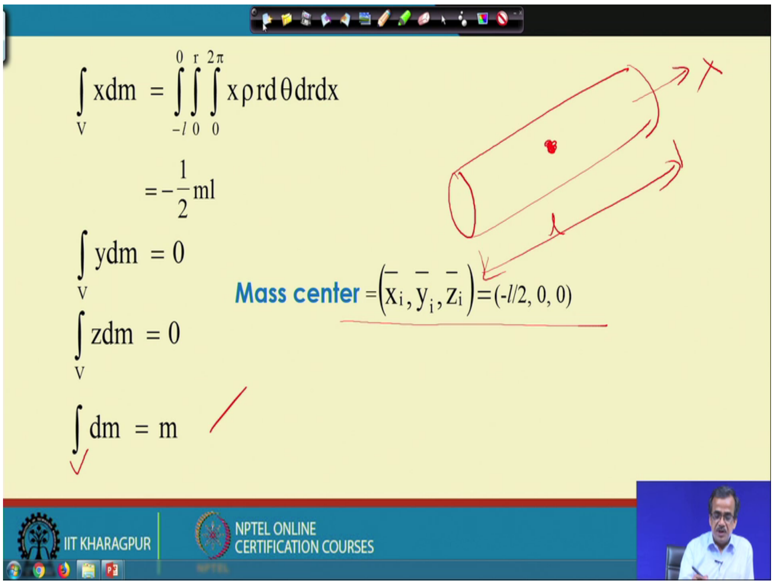

Now here, we have got this particular product of inertia, now volume integration xdm if you carry out. So, I will be getting minus half ml then integration volume integration ydm will become equal to 0, then volume integration zdm will become equals to 0. Now the mass center that is X i bar Y i bar Z i bar is nothing but minus l by 2 0 0 like if I draw this particular circular cross section that robotic link. So, my X direction is along this particular direction and the total length is l.

So, its mass center will be here whose coordinate is nothing but minus l by 2 0 0. So, this is nothing but the coordinate of the mass center. Then comes your integration the volume integration or over dm so is nothing but m. So, we can find out all such integration and this is a once again the volume integration next is. So, once you have got all such expression Now, I can find out the inertia tensor.

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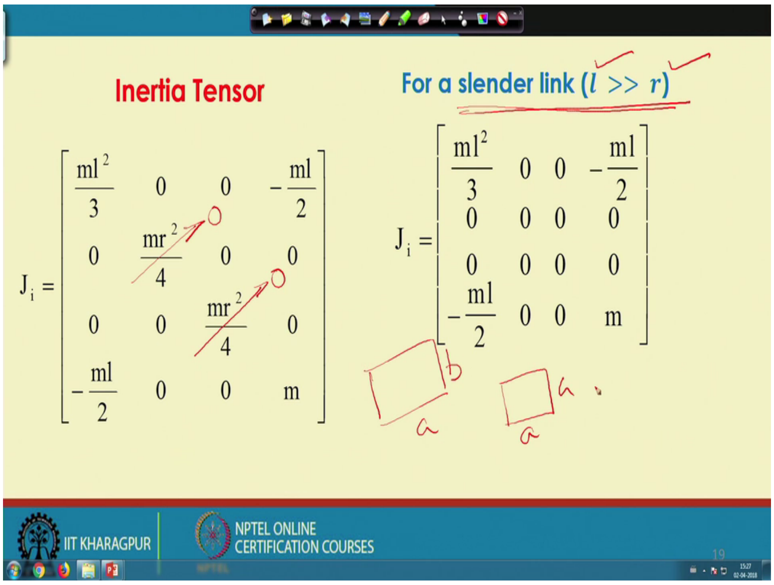

So, this inertia tensor for this particular link with circular cross section, that is denoted by J i will become equal to ml square by 3 0 0 minus ml by 2 then comes 0 mr square by 4 0 0 0 0 mr square by 4 0 minus ml by 2 0 0 m.

Now, if I consider the slender link the where r is very large compared to r, l is very large compared to your r in that case this mr square by 4. So, this can be neglected. So, this tends to 0, then mr square by 4 tends to 0. So, this will become the inertia tensor will become ml square by 3 0 0 minus ml by 2, then comes 0 0 0 0 0 0 0 0 minus ml by 2 0 m. So, this is nothing but the inertia tensor for the robotic link having circular cross section of radius r.

Now here, till now we have considered the rectangular cross section having the dimension a and b. So, I can also consider the square cross section like having the dimensions a and a. So, for this particular cross section robotic link we can also find out the inertia tensor. Now if you see the rho in robot we generally do not consider. So, this type of robotic link having the constant cross section. If you see in the robotic link the cross section is varying the same is true in our hand also.

For example say if I consider say this is the robotic links. Now if I take one cross section here if I take another cross section here the cross section is not the same. So, cross section is going to vary and it is having the varying cross section. So, determining this inertia tensor is not so easy, the same is true for the actual robotic link. In actual robotic link generally we do not consider the link with constant cross section and the cross section is going to vary along the length.

So, it is bit difficult to determine the inertia tensor. In fact, we try to take the help of some sort of finite element analysis to find out what should be that particular your inertia tensor. So, this is the way actually we try to find out the inertia tensor for the robotic link.

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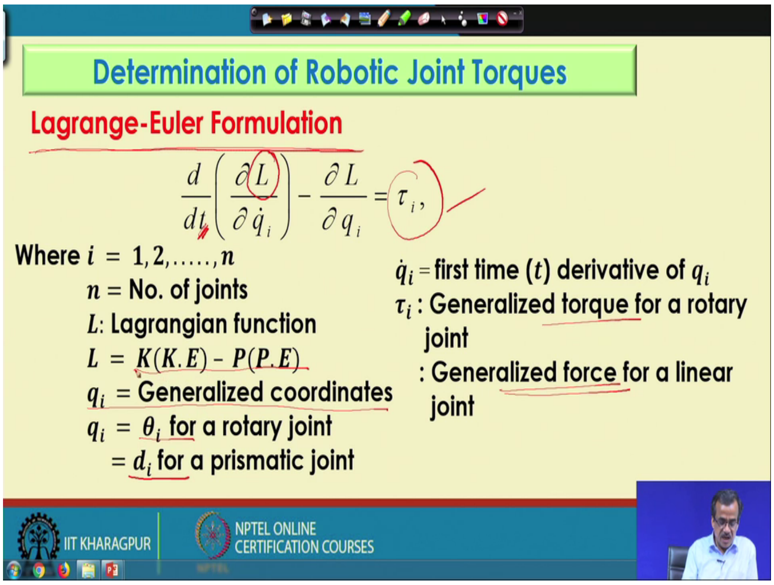

And once you have got the inertia tensor, now we are in a position to determine the expression the mathematical expression for the joint torque or the joint force.

Now, here actually we are going to use a Lagrange Euler formulation truly speaking this is the Lagrangian method Lagrange approach. Now the rule is or the equation is something like this ddt of partial derivative of L with respect to qi dot minus the partial derivative of L with respect to qi is nothing but your tau i. Now I am just going to define the different terms. So, here the T is nothing but time. Now L is nothing but the Lagrangian a Lagrangian of a system of a robotic system is nothing but the difference between the kinetic energy and the potential energy, that is nothing but the lagrangian of the robotic system.

Now, here this qi is the generalize coordinate for example, if it is a rotary joint. So, qi is nothing but theta i that is the joint angle if it is a prismatic joint that is nothing but the link offset that is di now qi dot is nothing but the first time derivative of qi and your tau i is nothing but the generalized torque if it is a rotary joint and if it is a linear joint, that is nothing but the generalized force. So, our aim is to determine the mathematical expression for this particular the tau. Now let us see how to determine the mathematical expression for this particular tau.

Now, to find out this mathematical expression, the first thing we will have to do is we will have to find out the expression for this particular the Lagrangian. And this Lagrangian is nothing but once again this is the difference between the kinetic energy and the potential energy; that means, we will have to determine the kinetic energy for the whole robot. And we will have to find out the potential energy and this particular difference of kinetic energy and potential energy is nothing but is Lagrangian.

So, my first task is to determine the Lagrangian of this particular the robotics system, now let us see how to determine the Lagrangian of this particular the robotic system.

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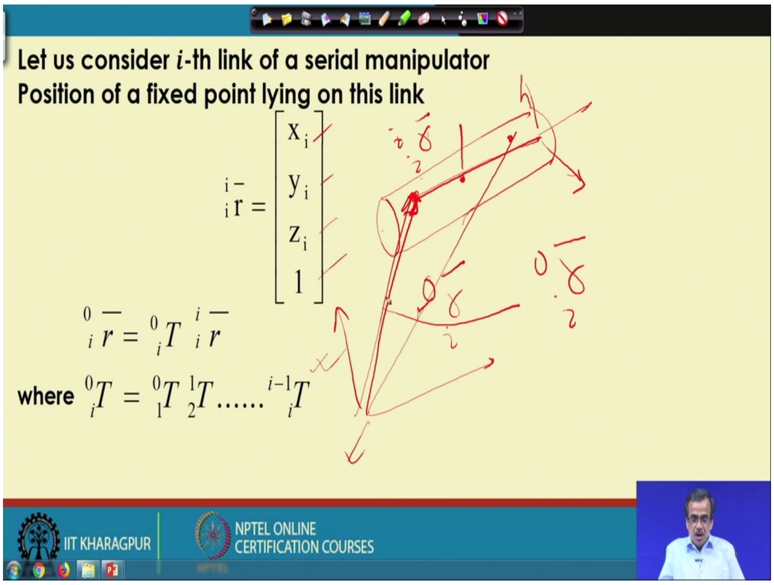

Now let me once again start with a particular the elements small element whose the coordinate in its own the coordinate system. For example, say if I consider just like the previously the way I consider, say I have got is a robotic link and here the coordinate system I am having here and the motor is connected here ok.

So, I am just trying to find out the mass center here and if I just consider a particular point or say this particular point. So, this point in this coordinate system is nothing but r i with respect to i. Now with respect to the base coordinate system. So, the same point I am just going to find out and that is nothing but is your r i with respect to 0. So, our aim is to determine. So, this particular r i with respect to 0, but in this particular point; so I am just trying to find out. So, this particular point not this; so this particular point with respect to the base coordinate system.

So, this is nothing but r i with respect to 0, and this I am trying to find out provided this is known that is a r i with respect to i. So, this r i with respect to i is nothing but X i Y i Zi 1 and this r i with respect to 0 is nothing but T i with respect to 0 multiplied by r i with respect to i. Now, this T i with respect to 0 is nothing but T 1 with respect to 0 T 2 with respect to one up to T i with respect to i minus 1 and this particular T is nothing but the transformation matrix.

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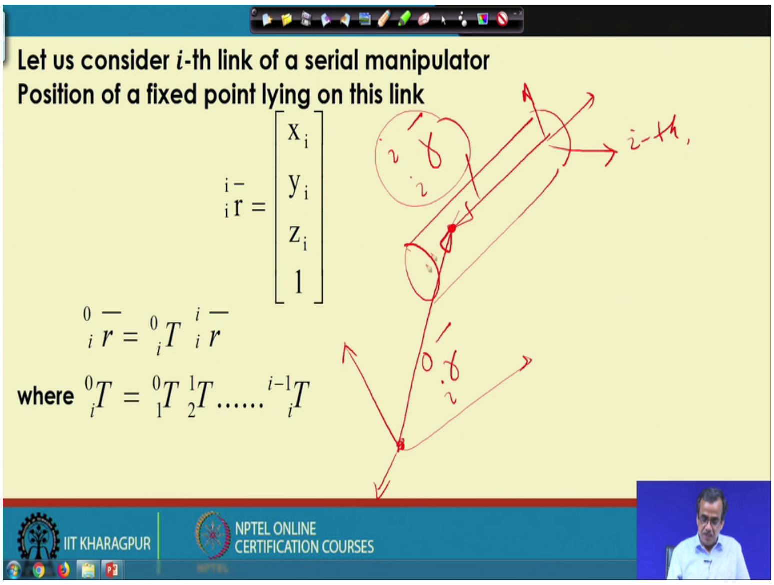

Now, once again let me let me repeat. So, so this is actually the circular link and my coordinate system is here ok. So, I have got a point here and with respect to this particular coordinate system. So, this dimension is nothing but r i with respect to i and I have got the base coordinate system here and the same point I am trying to find out and that is nothing but is your r i with respect to 0. So, r i with respect to 0 is nothing but T i with respect to 0 multiplied by this r i with respect to i and this is nothing but the i-th coordinate system and this is nothing but the base coordinate system.

So, in i-th coordinate system, this particular the position vector is known, now I am trying to find out r i with respect to the base coordinate point base coordinate frame the same point. So, this is the way actually we can represent your r i with respect to 0.

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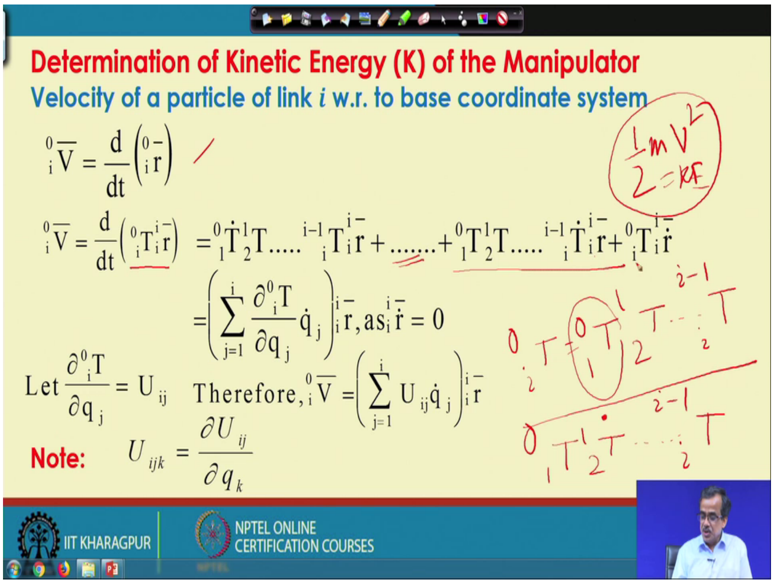

Now actually what I am going to do is, I am trying to find out the kinetic energy of the whole robot. Now if I want to find out the kinetic energy, the kinetic energy is nothing but half mv square.

So, the expression for the kinetic energy is nothing but half mass multiplied by V square. So, this is nothing but the kinetic energy. Now here actually what we are going to do, first we are trying to find out the kinetic energy of one small element lying on a particular link then we are trying to find out the kinetic energy for the whole links say i-th link and after that we will try to find out the kinetic energy for the whole robot or the whole robotic system.

Now, to determine the kinetic energy of the particle; so what you do is, you will have to find out the velocity of the particle. Now the velocity of the particle that is V i with respect to 0 with respect to the base coordinate frame. So, this is nothing but the rate of change of the position; that means, your ddt of r i with respect to 0. So, this r i with respect to 0 is nothing but the position of that particular the differential mass with respect to your the base coordinate frame, and a rate of change of that particular position with respect to time is nothing but V i with respect to 0; that means, the velocity of that particular particle with respect to your the base coordinate frame.

Now, this r i with respect to 0 as I told can be written as T i with respect to 0 multiplied by r i with respect to i. Now, this T i with respect to 0; so this can be written as T i with respect to 0 is nothing but T 1 with respect to 0 multiplied by T 2 with respect to 1 and the last term will be your T i with respect to your I minus 1 T stands for your transformation matrix.

Now, I will have to find out the derivative with respect to time. So, this T i with respect to 0 is nothing but this. So, if I find out the derivative with respect to time. So, first I will have to concentrate on this that is T 1 with respect to 0 dot multiplied by T 2 with respect to1 and there are a few terms, and the last term is T i with respect to i minus 1.

The next term will be like T 1 with respect to 0. So, here T 1 with respect to 0, then T 2 with respect to one dot then comes the last term will be your like T i with respect to i minus 1 ok. So, multiplied by r i with respect to I, and the last term will be this that is T 1 with respect to 0 T 2 with respect to 1. And the last term is your T i with respect to i minus 1 dot multiplied by r i with respect to i plus T i with respect to 0. So, this particular r i with respect to i dot. So, this is another term.

So, this is the way actually we can find out this particular the time derivative now let us try to concentrate on this particular term. Now if we concentrate on this particular term.

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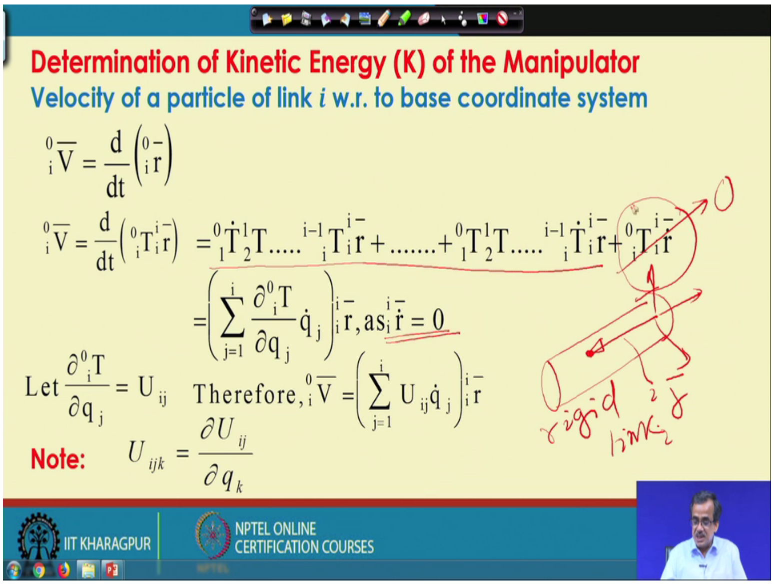

So, this r i with respect to i dot means what supposing that I have got one robotic link the rigid link something like this. Now, if this is the rigid link and I am just going to concentrate on a particular point ok. And this supposing that I have got the coordinate system here. So, this is actually the position of this particular the differential mass and this is nothing but r i with respect to i dot

Now, the rate of change of this particular position with respect to time will be 0, because this is a rigid link. So, this is a rigid link. So, for this particular rigid link your this r i with respect to i dot is nothing but is equal to 0. So, this particular term will become equal to 0. So, we are left with this up to this. Now here I just want to mention that if we consider robotic link like flexible robotic link, we cannot assume that this r i with respect to i dot is equal to 0. So, this will become nonzero for a flexible link and we have a few robot having flexible link also.

And determining this particular your V i with respect to 0 is not so, easy and we will have to consider. So, this particular your flexible link and this particular term is non-zero and to determine once again we will have to take the help of finite element analysis. Now here actually a what we do is so, if you can see. So, this particular term this particular term can be written like this in a short form.

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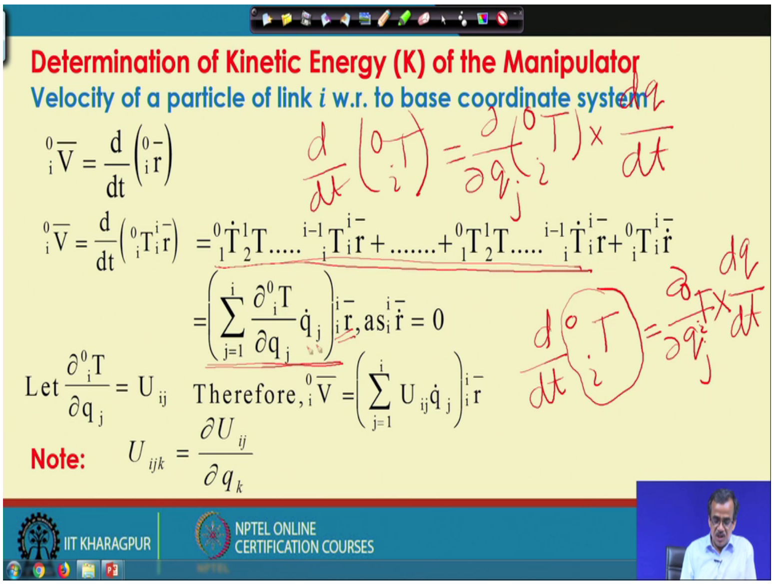

Now if we concentrate on this. So, that is your ddt that is the derivative with respect to time of your T i with respect to 0.

So, this is nothing but the transformation matrix. Now ddt of T i with respect to 0. So, this can be written as the partial derivative with respect to your q j partial derivative with respect to qj multiplied by d q dt. So, ddt of T i with respect to 0 is nothing but the partial derivative of this of T i with respect to 0 multiplied by dq dt let me write it once again. So, d dt of T i with respect to 0 is nothing but the partial derivative with respect to qj of T i with respect to 0 multiplied by is your dq dt.

So, it can be written something like this ok. So, this particular term has been written in this particular form that is your partial derivative with respect to qj of T i with respect to 0, and dq dt is nothing but qj dot and we have got this particular thing that is a r i with respect to i. So, this particular expression can be written in short form like this, and this is nothing but is your like vi with respect to 0; that means, the velocity of that particular particle with respect to the base coordinate frame is nothing but this particular the expression.

Now, here I am just going to use another symbol that is partial derivative of T i with respect to 0 with respect to your qj is nothing but U ij is a another symbol I am using; that means, your vi with respect to 0 can be written as summation j equals to 1 to i, then comes U ij. So, U ij is what; so this in place of this particular. So, I am using U ij multiplied by qj dot multiplied by r i with respect to i and here this U ijk is nothing but the partial derivative of U ij with respect to qk. So, these particular symbols we are going to write down just to write in a very compact form.

So, let us see how to write down this particular thing in a very compact form.

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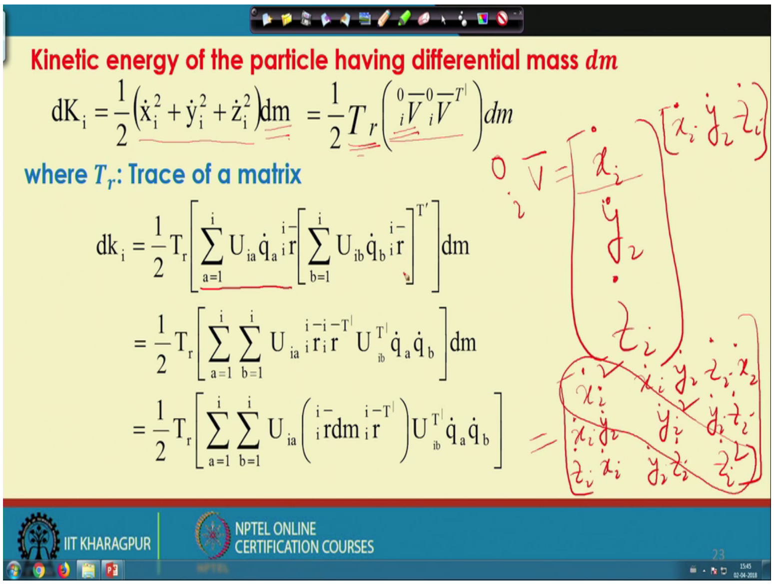

Now, kinetic energy of the particle having the differential mass is nothing but half mv square. Now here the mass of this particular differential mass is dm. So, half dm v square now here this particular velocity, which is a vector has got 3 components. So, this particular components velocity has got your 3 components like say xi dot then comes your yi dot and zi dot. So, these are the 3 components of this particular the velocity ok

Now here, so this the kinetic energy of the differential mass dm the small particle that is dKi is nothing but half dm, then xi dot square plus yi dot square plus zi dot square. Now this V bar that is nothing but a V i with respect to 0 is nothing but this and now this V i with respect to 0 trace of that. So, this is nothing but a 3 cross 1 matrix and I can write down actually here like its trace is nothing but xi dot yi dot then comes your zi dot now here if I just multiply.

So, I will be getting your first row first column that is xi dot square xi dot square, then first row second column xi dot yi dot then comes your zi dot xi dot then yx. So, I will be getting xi dot yi dot then comes your yi dot square then comes your yi dot zi dot, then comes your zx that is zi dot xi dot then comes your yi dot zi dot and then comes your zi dot square.

So, this type of like 3 cross 3 matrix we will be getting. Now here we try to concentrate only on the diagonal elements and this diagonal elements is nothing but trace of this particular the matrix ok. So, xi dot square plus yi dot square plus zi i dot square and that is a actually nothing but trace of. So, trace of. So, this particular V i with respect to 0, V i with respect to 0 transpose. So, trace of that is nothing but X i dot square plus Y i dot square plus Z i dot square dm.

Now, this dKi is nothing but your half trace of this vi with respect to 0 now we have already derived that V i with respect to 0 is nothing but summation a equals to 1 to i U ia qa dot r i with respect to I this I have already seen and then. So, this particular V i with respect to 0 T prime this is nothing but is your summation b equals to 1 to i Uiv qv dot r i with respect to i trace of that dm. Now here, I one thing I just want to mention now here this summation I have taken a equals to one to i. And here I have taken a v equals to 1 to i and this I have done very purposefully; for example, say if I consider. So, it is a rotary joint.

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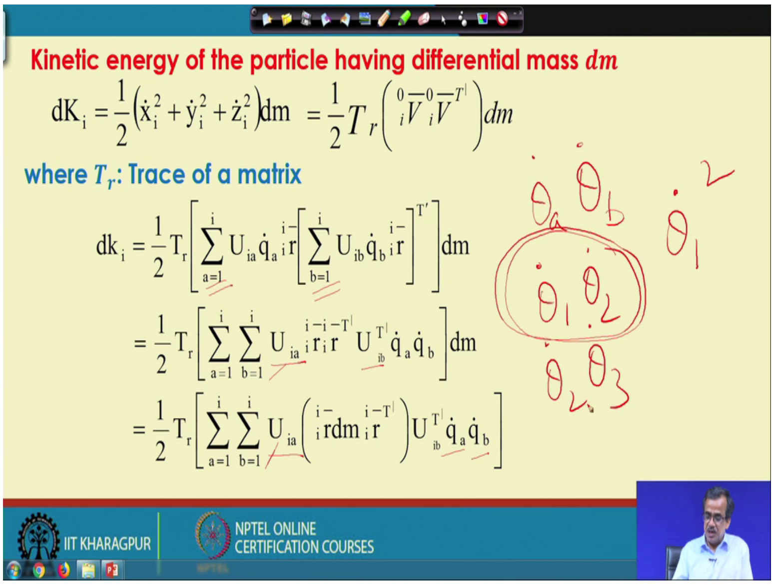

So, in place of this qa dot I am just going to write down theta a dot and in place of this qb dot I am just going to write down like qb dot. Now if you see the final expression which I am going to derive. So, there is a possibility there will be a few terms like theta 1 dot multiplied by theta 2 dot then might be theta 2 dot multiplied by theta 3 dot something like this ok. Now if I do not consider the separate range for this particular summation. So, I am just going to miss this particular the combination of theta one dot multiplied by theta 2 dot.

Now, if I consider both are a equals to 1 to i and b equals to 1 to i. So, there is a possibility I will be getting theta 1 dot square. Now for a very special case I will be getting theta 1 dot square and where a will become equals to b, but just to keep this particular possibility alive. So, I have taken 2 separate range for this particular the summation. Now, once you have written a like this we can a rearrange in this particular format half trace summation a equals to 1 to i summation b equals to 1 to i U ia r i with respect to i, r i with respect to i transpose.

Then comes U i be a transpose then q a dot qb dot dm this can be further rearranged in this particular format half trace summation a equals to 1 to i, summation b equals to 1 to i, qa qi a. Then, this r i with respect to i dm, r i with respect to I transpose of that then U ib transpose qa dot qb dot. So, it can be rearranged in this particular the format.

Thank you.