Mechanism and Robot Kinematics
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Lecture – 38
Coordinate Transformation– I

When we analyze multi degree of freedom systems like robots, we need to deal with multi input and multi output systems. Now, in a robot manipulator, as opposed to in a mechanism, in the constraint mechanism, we have various links moving in space or in a plane on manipulator various links moving in 1 plane. And they can change their position and orientation as they move.

Now, it is a very important thing in mechanics in general and kinematics, in particular to be able to track a point in a moving frame. So, for example, we always talk of this problem, that, suppose, there is a satellite or an aircraft which is in space which is flying and there is certain point which is moving with respect to the aircraft or with respect to the satellite, then the representation of motion of that point or particle is very easy in the aircraft frame or the satellite frame.

But, suppose, I want to represent this same motion in the earth fixed frame, then it becomes a little complicated. Now, for this, we use frame transformations, that simplifies the way we can relate these 2-this representation of vectors in a frame that might be moving in certain body which might be moving. So, we need to understand coordinates coordinate transformations in order to be able to represent a point or a particle or maybe another link, which is mounted on a moving link.

So, in this lecture, we are going to start discussions on coordinate transformation and we will see how this comes these concepts are used in very easily analyzing robot manipulators. In this in these lectures, we are going to restrict ourselves to the planar case. Once the planar case becomes clear, then going over to the special case of the 3-dimensional case becomes very simple. It is just a one stepead.

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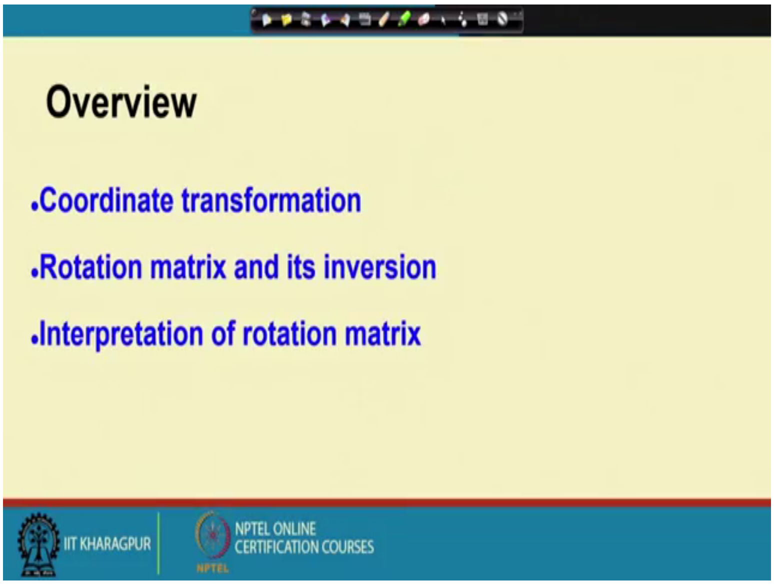

So, the overview of this lecture is that, we are going to first start with the coordinate transformation, we are going to look at rotation matrix and it is inversion and we are going to interpret the rotation matrix, so that, we are able to understand the elements that construct the rotation matrix.

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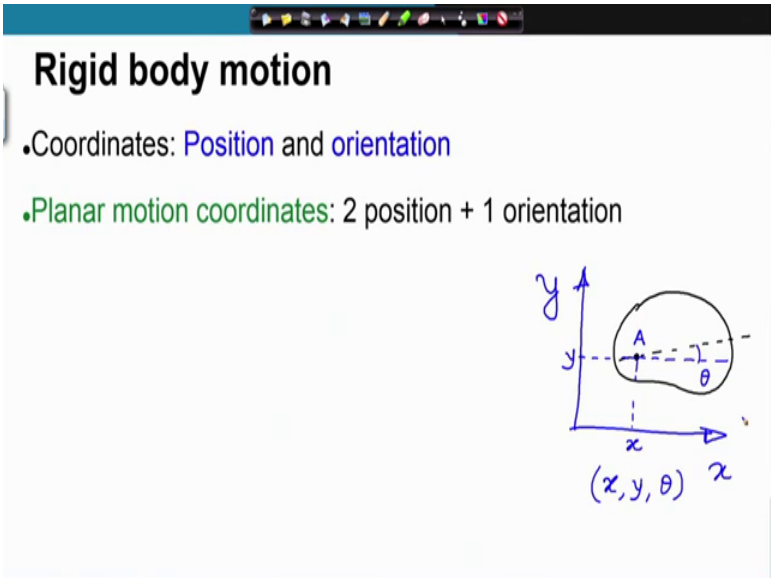

Now, rigid body motion is tracked by certain coordinates. So, how do you track a body moving in space a rigid body moving in space; or on the on the plane?

So, we have coordinates which are the position and orientation coordinates of the body. So, the position coordinates can be xyz and the orientation coordinates can be rotations about the 3 axis. There are various ways of representing orientation. Here, we are going to only restrict ourselves to planar orientation. So, the in planar motion coordinates, we have 2 positions and 1 orientation.

So, let me show you what I mean. So, in the plane, let us say, the xy plane, if I have a body, so, this dashed line is fixed to the body. Then, by tracking the xy of this point, let us say A on the body and the orientation through this angle let us say theta. So, this angle theta is the angle measured between the body fixed line and the x axis. So, we have these 3 coordinates x y theta. So, by keeping track of x y and theta, I can uniquely locate the body in a plane. So, we have 2 positions and 1 orientation coordinate for planar motion.

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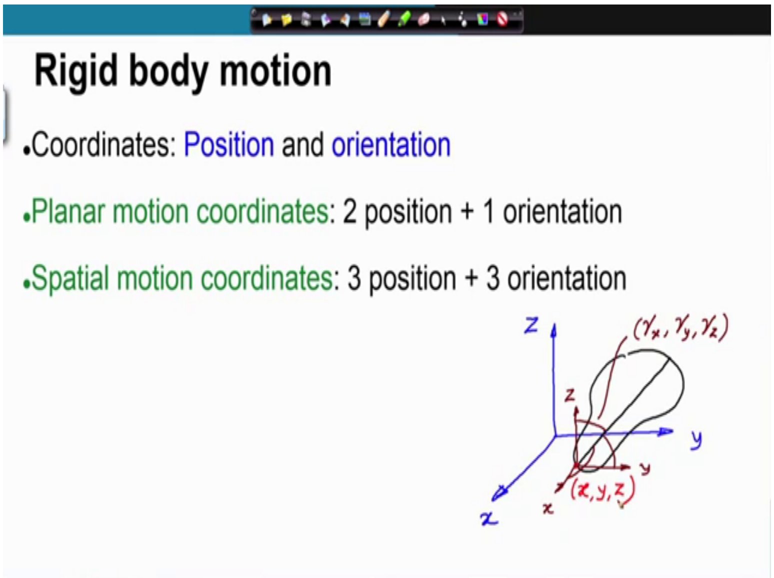

For spatial motions, we require 3 position coordinates and 3 orientations coordinates. Suppose, I have a body in which a line is fixed, then I can locate coordinates of a certain point on the body and the orientation by looking at possibly the direction cosines, these angles, that the line makes with the local xyz coordinate system, which is parallel to the blue coordinate system.

So, by having this these angles, let us say, gamma x, gamma y, gamma z, I can fix the position of the body. So, x, y, z and gamma x, gamma y gamma z, these will fix the position and orientation of the body in space. So, I and acquire 6 coordinates for a body in space.

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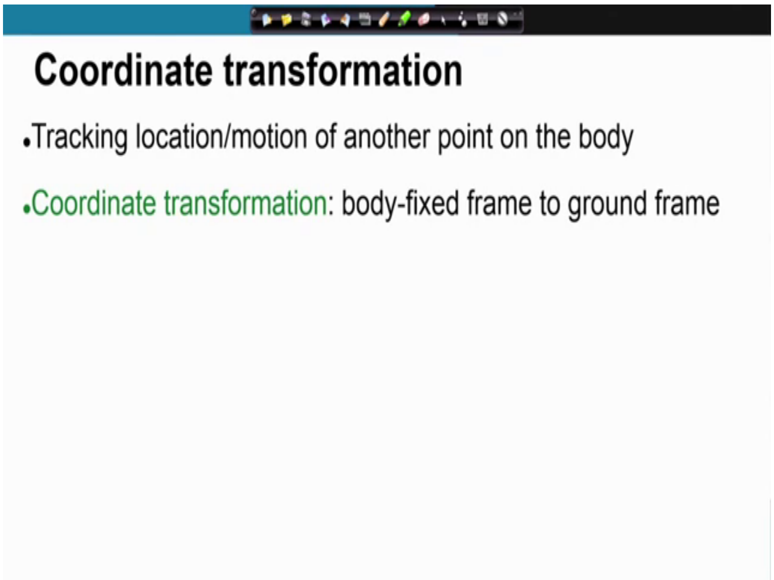

Now, in order to track the location or motion of a point on a body which is in motion may be in play on the plane or in space. So, if I have a body which is moving in space, let us say, and there is something else which is also moving with respect to that body. I would like to find out the coordinates of that point in my earth fixed frame. So, this is what I want to find out. So, I want to track the location on motion of another point on the body. So, for that I require coordinate transformation. So, this coordinate transformation, we will take the coordinates of the point in the body fixed frame to the ground frame. So, that is the purpose of the coordinate transformation.

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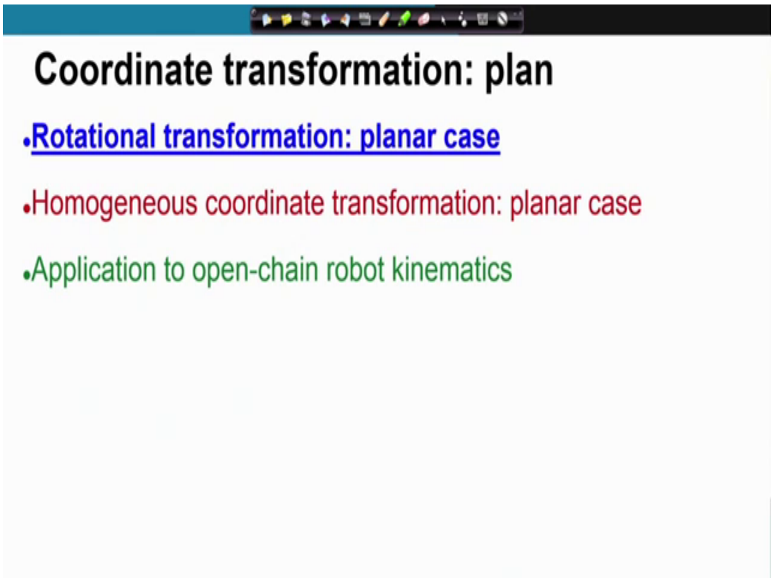

Now, our plan for coordinate transformation I have listed out here. So, we will begin by looking at the rotational transformation; the planar case, then we will look at the homogenous coordinate transformation in the in the plane and we will finally, we will look at applications of the concept of coordinate transformation for analysis of robot open loop open chain robots.

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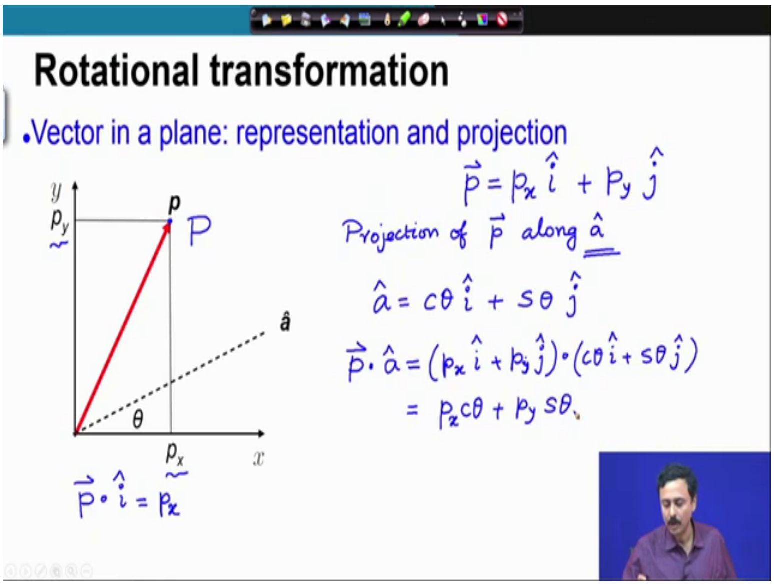

So, we will start with discussing the rotational transformation. Let us consider a vector in the plane. As I have shown you here, so this px is a vector. So, a P is a vector in the plane with coordinates px and py. So, normally, we tend to represent this vector p as p x i cap plus py j cap. Now, this P is a position vector suppose and suppose, it is a position vector of a point P. So, the vector p is a position vector of a point P.

Let us consider a direction. Here, I have represented as a cap which is at an angle theta. I would like to find the projection of the vector p along a cap. So, I want to find out the projection of the vector p along the unit vector a cap. Now, you already know and we have also discussed the concept of projection. For example, if I want to project the vector p along x axis then, so if I want to project vector p along x axis, then I take a dot product with i cap and that gives me p x.

Now, our problem is little more general. We have a unit vector a cap, which is the direction at an angle theta with respect to the x axis. Therefore, I can very easily represent the unit vector a cap. So, cosine theta, I am writing c theta for cosine theta i cap plus sin theta I will write s theta for sin theta times j cap. So, that is the unit vector a cap.

So, therefore, projection of p along a, is nothing but the dot product of p vector with a cap. So, this is the p vector and a vector is and that gives us p x cosine theta plus p y sin theta. So, that is the dot product of p with a cap. And this is this concept is what we are going to use for constructing the rotational transformation. So, let us see, how we do that.

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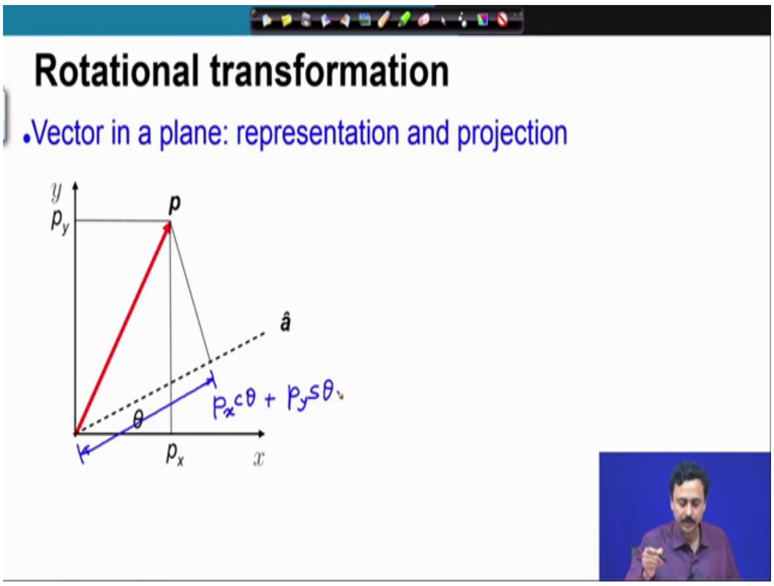

So, this projection, so this distance, is nothing but what I just now wrote out p x cosine theta plus py sin theta. So, that is the projection and this is what we are going to use.

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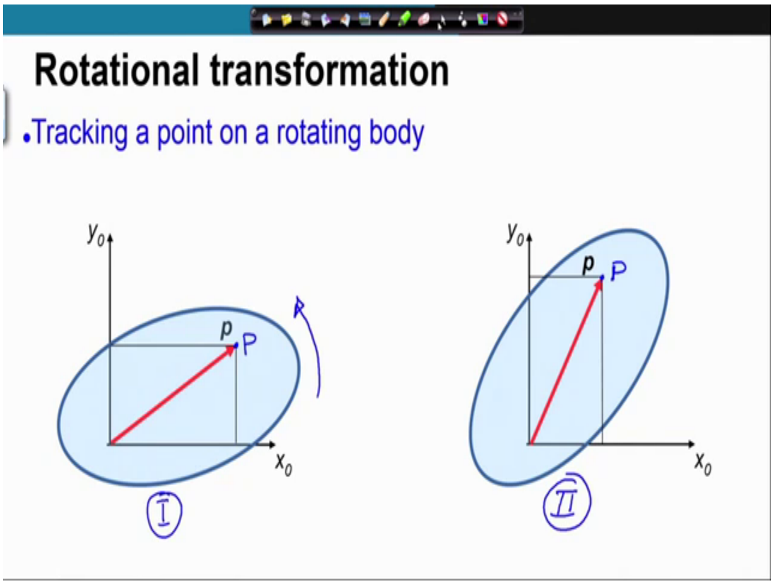

So now, we are going to directly go into the problem of tracking a point on a rotating body.

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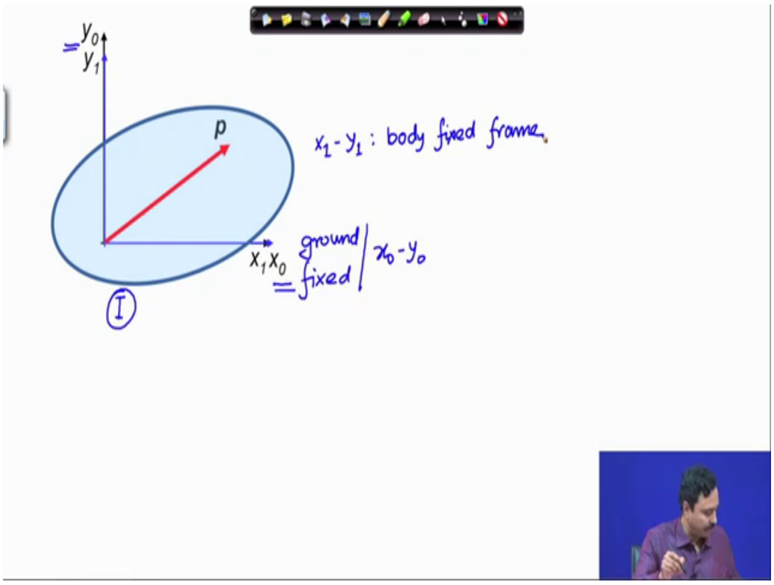

So, here, I have shown you a body in position 1 which rotates and goes over to position 2. There is a point P on the body, whose position vector is this vector p.

So, this point p goes over here, upon rotation. And the new vector is again represented here as p. So, p vector represents the position of the point P. Here, I have shown you the body in configuration 1. So, this is configuration 1 and I have also shown 2 frames. Here is 1 frame x 0 y 0 which we will fix this is ground fixed. So, x 0 y 0 is ground fixed. The x 0 y 0 frame is ground fixed, whereas, the frame x 1 y 1 is body fixed. So, the frame x 1 y 1 is body fixed and x 0 y 0 is ground fixed. What happens when we rotate the body ?

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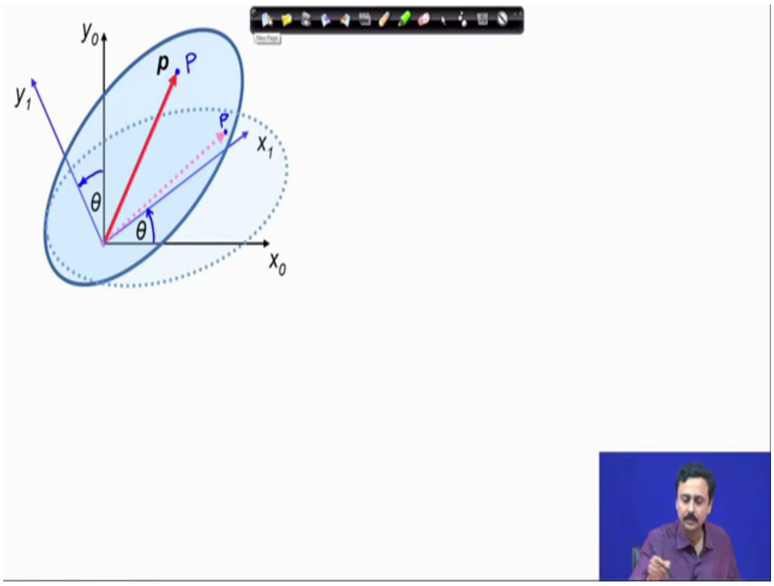

X 1 y 1 has rotated by this angle theta as you can see. This was the initial position of point P. And now, the new position of point P is here. Our objective is to locate the new point P in x 0 y 0. Here, I have removed the original position of the body in configuration 1. So, we can only see the body in configuration 2. The body fixes frame x 1 y 1 and the ground fixed frame x 0 y 0. We can also see the new location of point P.

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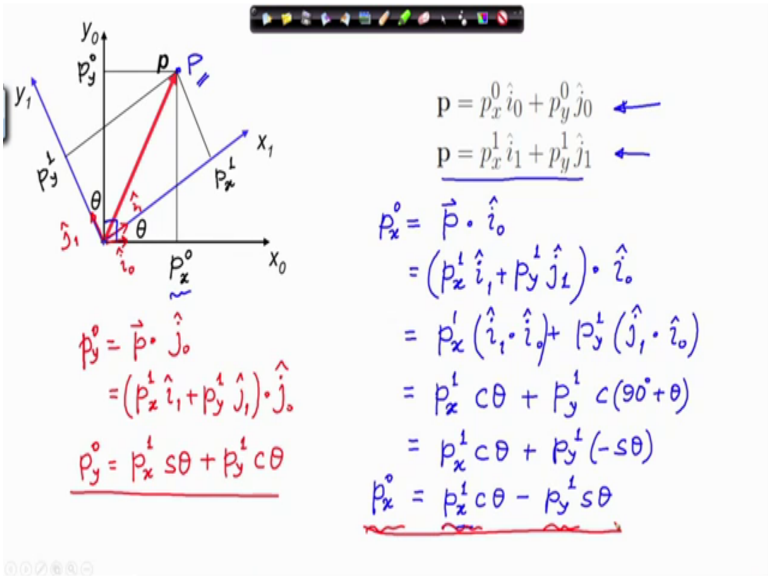

So, this is the new location of point P. What I want to determine is, this p x represented in frame 0 and p y represented in frame 0. In terms of p x represented in frame 1 and p y represented in frame 1.

Therefore, I have these 2 representations of the vector p. Remember that, it is the same vector p, which is the position vector of point P; P. In the ground fixed frame, the representation of the vector p is px 0 i 0 cap plus py 0 j 0 cap, where i 0 cap and j 0 cap are unit vectors along x 0 and y 0. And similarly, in frame one the representation of the same vector p is px 1 i 1 cap plus py 1 j 1 cap.

Remember, this is the same vector p being represented in 2 different frames. Nothing else. And we would like to find out the relation between this px 0 py 0 and px 1 py 1. This is what our objective is. Now, how do we go about doing this? I want to find out px 0. So, how do I find out px 0 px 0? Is nothing but the vector p dot i 0 cap; that is, px 0. Now, p has this representation in the new frame, like this and this dot i 0 cap. So, this becomes p x 1 i cap 1 dot i 1 cap dot i 0 cap plus p y 1 j 1 cap dot i 0 cap. So, this becomes px 1.

Now, what is i 1 cap dot i 0 cap? i 1 cap dot i 0 cap. Remember that i 1 is i 1 cap is along x 1 and i 0 cap is along x 0. So, what is the dot product of 2-unit vectors which has angle theta between them? Is nothing but the cosine of the angle between the 2 vectors? So, this must be cosine theta plus p y 1.

Now, j 1 cap dot i 0 cap. So, j 1 cap is nothing but a vector a unit vector along y 1 now, this unit vector j 1 cap I have drawn here dot i 0 cap. So, it must be cosine of the angle between i 0 cap and j 1 cap. Now that angle is 90 plus theta as you can see. You have 90 degree plus theta that is the angle between i 0 cap and j 1 cap. So, this becomes cosine of the angle between the 2 which is 90 degree plus theta. So, that becomes p x 1 cosine theta plus p y 1.

Now, cosine of 90 plus theta is nothing but minus of sin theta. So, therefore, this becomes p x 1 cosine theta minus p y 1 sin theta. So, this is px 0 in terms of so, this is px 0 in terms of p x 1 and py 1. So, we have related we have one relation. Now, we obtained one relation between px 0 px 1 py 1 in terms of the rotation of the frames theta.

Now, following this procedure, I can find out also py 0. The procedure remains the same; which means, I must have py 0 as p dot j 0 cap. And then, once again, I use the representation in frame 1 carry out the dot product write out the angles between the unit vectors and I will get a second relation. So, that will turn out to be I will have i 1 dot j 0 i 1 dot j 0 that is the angle. So, because both are unit vectors.

So, it is the cosine of the angle between i 1 and j 0. So, that will be cosine of 90 minus theta, which means, this will be sin theta. And then I will have py 1 times j 1 dot j 0 j 1 dot j 0 is the angle between j 1 and j 0 cosine of the angle between j 1 and j 0 and that is again theta. So, this will become cosine theta this you can easily check.

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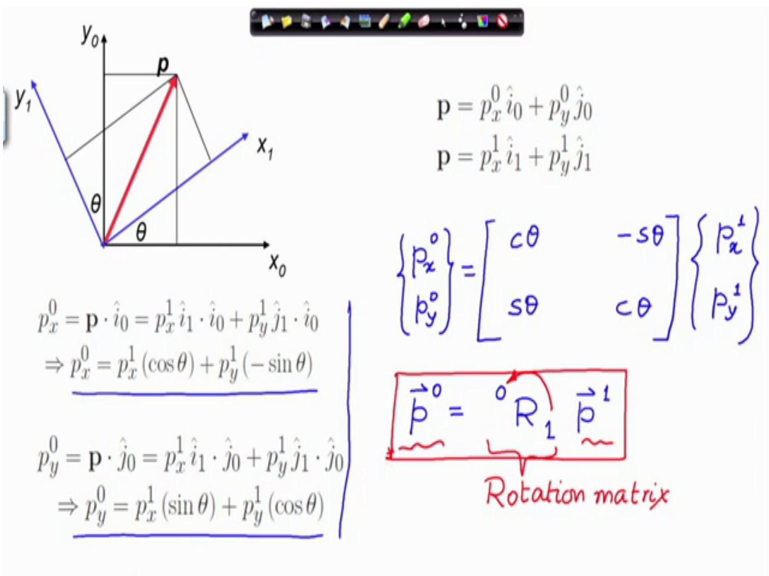

So, I have finally, these relations. So, let me show these things formally. So, I have these relations. I have shown you how to obtain these relations. Now, I will assemble them px 0 py 0 I will write as a column vector is equal to some matrix times p x 1 p y 1. Because I have these 2 relations, on the left we have px 0 py 0 and on the right, we have px 1 py 1. So, I have written them as matrix times px 1 py 1 and if you fill in the elements of these of this matrix from the individual relations you will obtain this matrix.

Therefore, I will write this in a compact form as p vector represented in frame 0 as the rotation matrix which takes me from frame 1 to frame 0 times p represented in frame 1. So, this is how we are going to represent these vectors and show in which frame they are represented in. So, this p 1 is represented in frame 1 p 0 is represented in frame 0 and they are related through the rotation matrix R from 1 to 0. So, we will always say it goes from 1 to 0. So, on the right-hand side, I will x vector represented in 1 in frame 1 and on the left, I will get a vector represented in frame 0. So, this is the implication.

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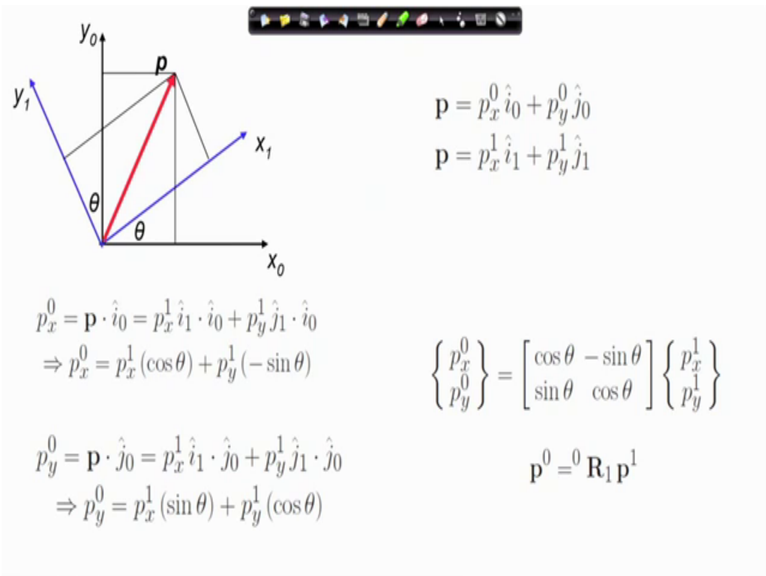

So, just to show this thing formally.

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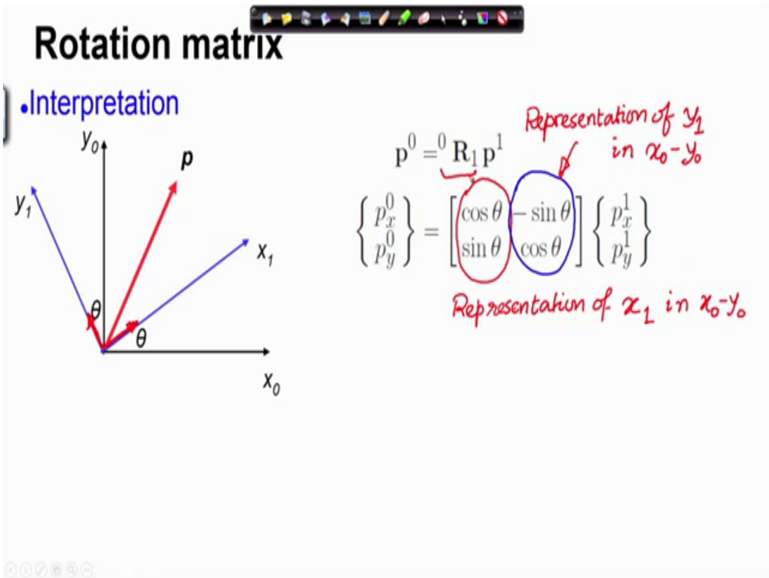

Now, we interpret this rotation matrix. If you look at column 1 of the rotation matrix, the column 1 you can very easily see is the representation of the vector x 1 or the axis x 1 in x 0 y 0.

The column 1 is the unit vector along x 1 being represented in x 0 y 0. So, this unit vector being represented in x 0 y 0 is nothing but that first column of the rotation matrix. The second column is the representation of the unit vector along y 1 in x 0 y 0. So, I will write this is the representation of y 1 in x 0 y 0 frames.

Now, if I interpret that this way, then it is very easy to construct the rotation matrix. I need not go through the steps that I have just now shown you. I can directly construct the rotation matrix. So, remember that, the 1st column of the rotation matrix that takes me from 1 to 0 is nothing but the representation of the frames x 1 y 1 in x 0 y 0. So, the first column is representation of x 1 in x 0 y 0. The 2nd column is the representation of y 1 in x 0 y 0. So, that will construct the rotation matrix 1 to 0.

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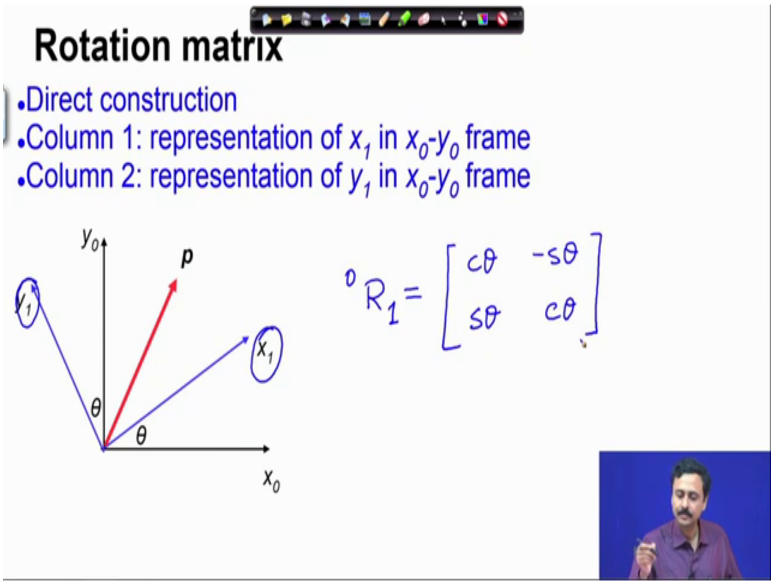

So, we can directly construct the rotation matrix. Let me do that. So, I say, that I want to construct this rotation matrix from 1 to 0 that will take me from frame 1 to frame 0, then I must represent the individual axis of the frame 1 in frame 0. So, frame 1 the first axis is x 1. So, let me represent x 1 in x 0 y 0.

So, this will be cosine theta sin theta and then, the 2nd column is the representation of y 1 in x 0 y 0. So, that is minus sin theta cosine theta.

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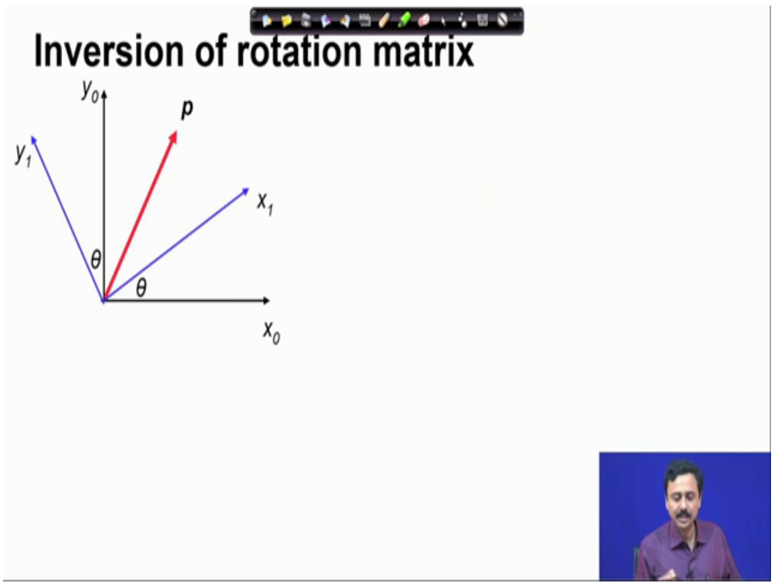

Now, when we invert the rotation matrix, is just the opposite thing that you do, you start with the vector representation in the 2 frames.

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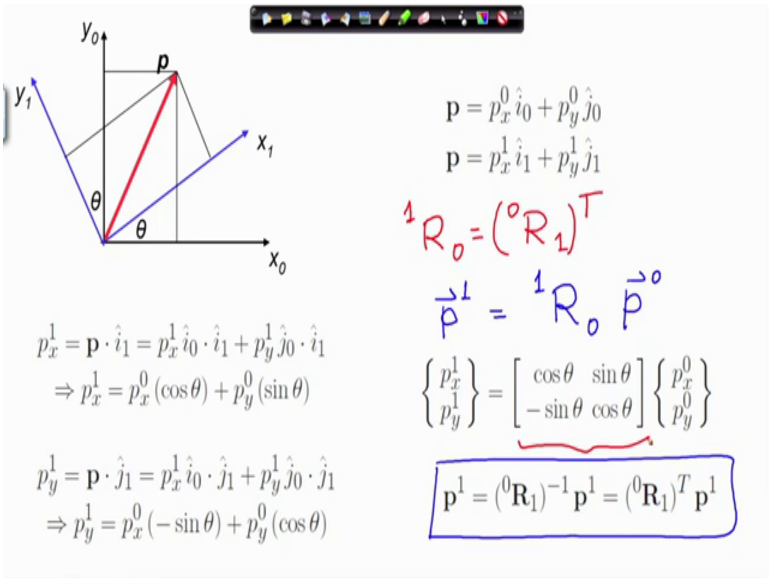

now you do a dot product with i 1 cap to get px in frame 1. So, that is the only difference.

Now, if you proceed this way, similarly, we will get py 1 and therefore, finally, you can represent this vector is p in frame 1. So, therefore, this must be the rotation matrix that takes me from frame 0 to frame 1 and p that must be multiplied by the vector p in frame 0 and that is being represented here. You can very easily check that R from 0 to 1 is the transpose of R from 1 to 0. R from 0 to 1 is nothing but R from 1 to 0 transpose and that expression you have here.

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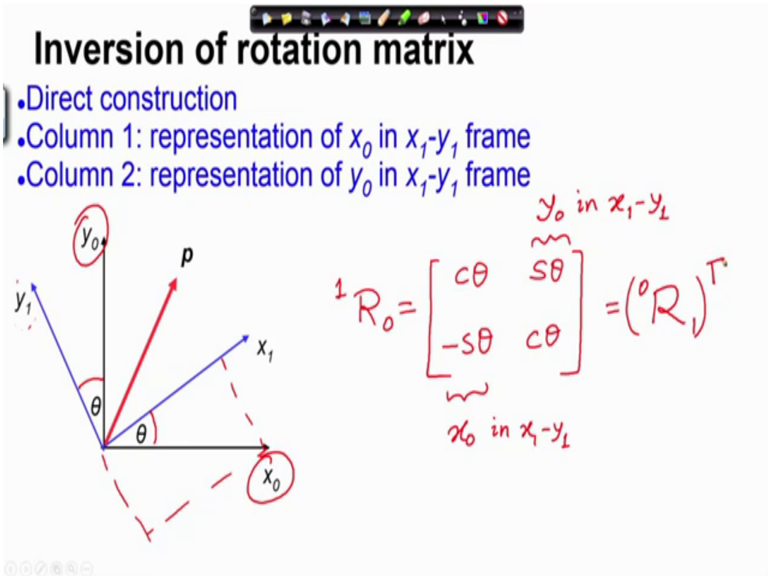

So, therefore, once again a direct construction let us look at it in direct construction of R from 0 to 1. So, what do I now have to do? I have to represent the frame x 0 y 0 in x 1 y 1. So, frame the axis x 0 the first column will be axis x 0 in x 1 y 1. So that, you can easily check is cosine theta this angle being theta. So, cosine theta and minus sin theta that represents x 0 in x 1 y 1.

The second column is representation of y 1 representation of y 0 in x. So, the second column is representation of y 0 in x 1 y 1. So, y 0 in x 1 y 1, now this being theta, so, I must have sin theta and cosine theta this is y 0 in x 1 y 1. So, that constructs the rotation matrix R from 0 to 1 and that you can check is the transpose of the matrix R from 1 to 0.

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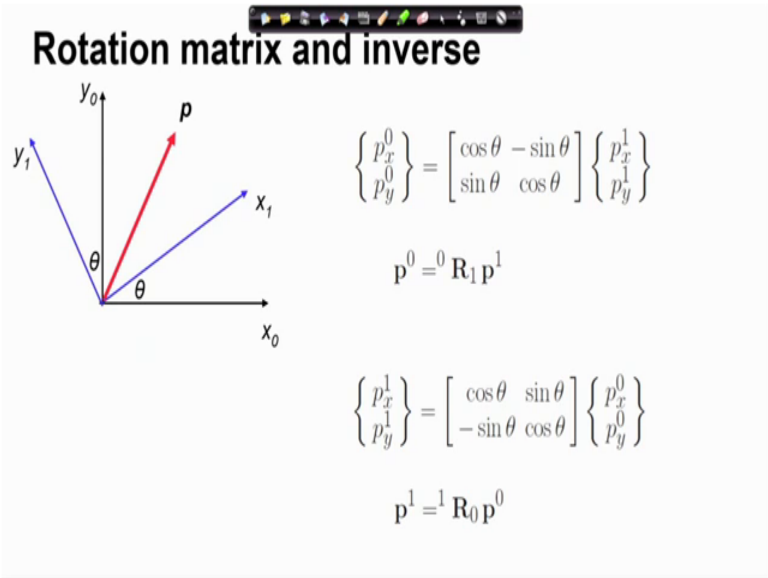

So, therefore, here I have written out the direct and the inverse rotation matrices from frame 0 to 1 and 1 to 0.

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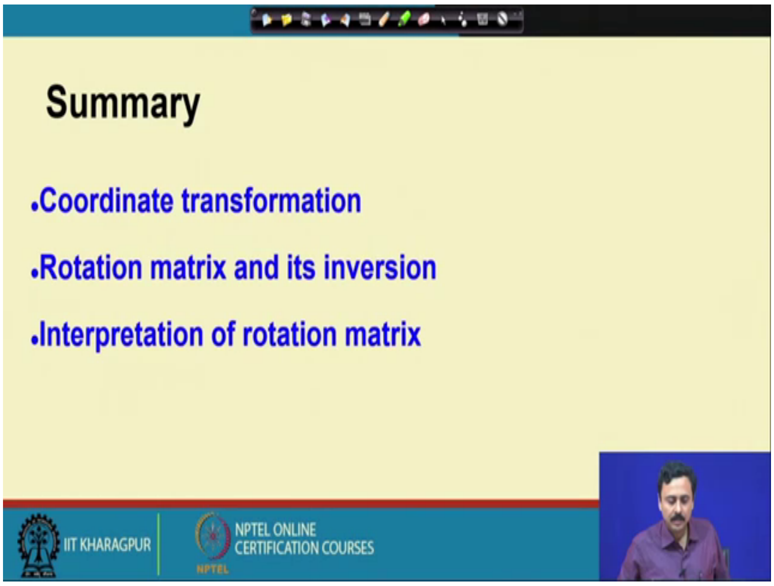

So, let me summarize what we have discussed in this lecture. We have looked at coordinate transformations, we have looked at the basis of coordinate transformation and why we need it. I have constructed I have shown you how to construct the rotation matrix and I have also given you the interpretation of the rotation matrix. And this interpretation helps us to directly write the rotation matrix by looking at the coordinate frame orientations. And we are going to use this rotation matrix in our subsequent discussions. I will close this lecture at this point.