Introduction to Fluid Mechanics
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Lecture- 54
Navier-Stokes equation- Part-IV

So, we will consider a second example of exact solution of the navier stokes equation through something which is very important in fluid mechanics. This is called as Couette flow. Consider 2 parallel plates with relative motion that means.

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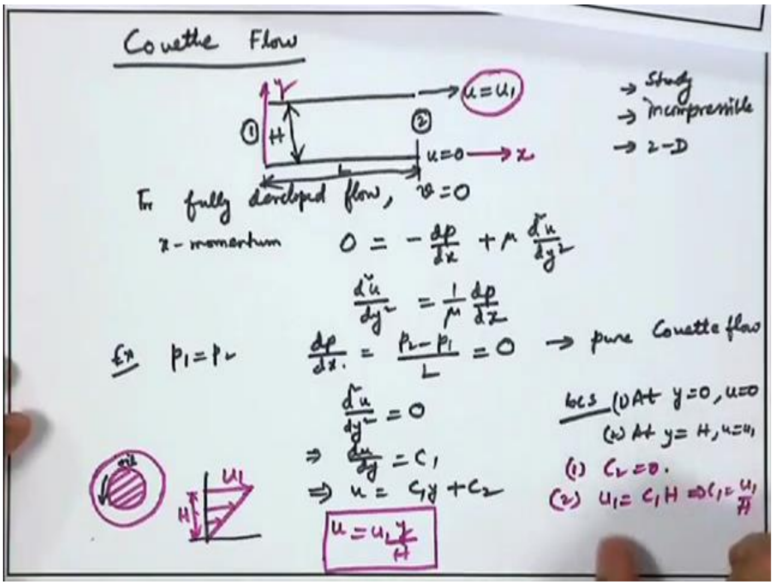

Let us say that the bottom plate is stationary and the top plate is moving toward the right with a velocity u 1. The gap between the plates is small and the flow is fully developed. Now you may ask the question that why are we studying this. Have you ever seen somebody pulling plate on the top of another plate? I mean I have never seen this kind of a situation.

So, if you now try to guess that still even though you have more practical situation of a pressure driven flow which is the Hagen poiseuille or plane poiseuille flow, the flow between 2 parallel plates is plane poiseuille flow that we have just studied, then why you are studying this Couette flow. I will come to the answer in a moment and this answer is very important. See very often we teach our student something without motivation. We give an example well we consider 2 parallel plates one plate is moving another plate is stationary, but students are not able to find out any relevance.

So, it is important that students should try to get a feel of why this kind of flow is important. Let us try to find out the velocity profile for this, so as usual for fully developed. So, we consider steady flow incompressible flow 2 dimensional flow. Now for fully developed flow, let us fix up our x and y axis also that will help in solving the problem, for fully developed flow you have v equal to 0 and x momentum equation, 0 is equal to minus d p d x plus mu d 2 u d y 2. So, d 2 u d y 2, is 1 by mu d p d x. Let us call this as section 1 let us call this as section 2.

Example, when p 1 is equal to p 2. So, d p d x, if this is the length of the channel is equal to p 2 minus p 1 by l, this is equal to 0. This is called as pure Couette flow; that means, there is no pressure gradient which is acting on it. Couette flow can also have a pressure gradient acting on it, but if there is no pressure gradient acting on it, it is called as pure Couette flow. So, far pure Couette flow, you have this is equal to 0; that means, d u d y is equal to a constant c 1; that means, u is equal to c 1 y plus c 2. What are the boundary conditions? At y is equal to 0, u equal to 0, this is the first boundary condition. And second boundary condition at y equal to h u is equal to u 1.

So, at applying the first boundary condition at y equal to 0 u equal to 0; that means, c 2 is equal to 0. And second boundary condition at y equal to h, u equal to u 1. So, c 1 is u 1 by h; that means, the velocity profile is u is equal to u 1 into y by h. So, velocity profile is linear, if the velocity profile is linear what is the rate of strain or rate of shear, the rate of strain or the rate of shear is what is d u d y. So, what is d u d y? D u d y is u 1 by h. So, by specifying the value of u 1, you can impose a particular value of shear. So, this is known as shear driven flow.

Just like you have pressure driven flow, this type of flow is also important and this is known as shear driven flow. So, the Couette flow is a typical example of something which is called a shear driven flow. Now what do you get such an example. Let us say that in industry we commonly get shafts, which transmit power. Now there is an outer casing within which the shaft is supported and that is called as bearing. So, this is the outline of the bearing. Now to avoid metal to metal contact between or material to material contact, in general between the shaft and the bearing there is a lubricating oil that is kept in the gap.

So, this lubricating oil is a thin layer, it is a very narrow gap. And then the shaft is rotating with a particular angular velocity whereas, the bearing the is stationary. If the gap between the shaft and the bearing is very small, this curvature effect can be neglected then these 2 can be thought of approximately has 2 parallel plates and one plate relative moving relative to the other. So, the fluid dynamics in bearings, if you want to understand that then the Couette flow can be one of the very basic mechanisms which helps us to understand this. There is also another motivation. And that motivation is very settle.

So, if you recall that what is the velocity profile in a pressure driven flow. So, if you consider the previous example, what was the velocity profile? It is a parabolic velocity profile. Now small part of a parabola is like a straight line. Let us say there is a biological cell which is sitting on the wall of this channel and we are interested to study the force exerted by the fluid on the cell. Then very close to the wall since we are considering only a small part of the parabola we can linearize it and we can consider a linear velocity profile that is acting on the cell depending on the dimension of the channel. Of course, if the channel itself is of the comparable dimension as that of the cell; then no question of this considering this small linear part.

But let us say the channel is 1 meter and the cell is 10 microns. So, for practical purposes this 10 micron velocity variation; velocity variation within this 10 micron is approximately giving a linear velocity profile. So, a linear velocity profile or a share driven flow is very practical. It is not always the question of one plate pulled over the other, that the philosophical perspective of creating a relative motion between the 2 plates that you have to consider.

So, the velocity profile if we draw, now this is u 1 and this is h now. So, far, So, good, but what happens when both pressure gradient and driving share acts on the system that plates are moving and p 1 minus p 2 is non 0. Let us work out a problem highlighting such a case.

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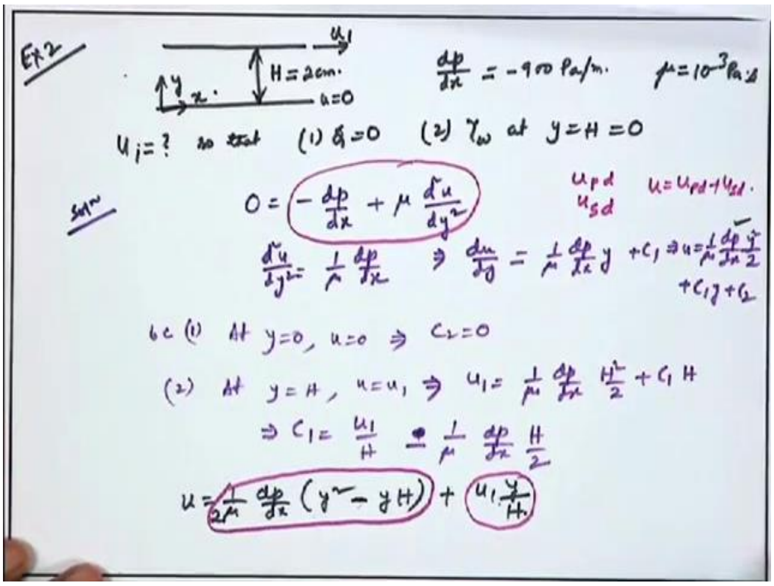

So, example 2, please open page number 7 of your lecture notes and you will find example 2. Consider steady incompressible fully developed flow of water at 20 degrees centigrade between 2 horizontal parallel plates with the gap between the plates is 2 centimeters. A constant pressure gradient d p d x is equal to minus 900 Pascal per meter drives the flow.

In addition, the upper plate moves with a uniform speed whereas, the lower plate is stationary. Find the velocity of the upper plate what should be u 1. So, that number 1, Q is 0 and number 2, tau all at y equal to h is equal to 0. The viscosity of water at 20 degrees centigrade is 10 to the power minus 3 Pascal second that is given. So, this is y and this is x. So, let us work out this problem. So, in this problem, you have both driving pressure gradient as well as shear. Now for steady 2 dimensional fully developed in compressible flow all those assumptions being valid. Let us write the governing equation, so solution.

What is the governing equation? 0 is equal to minus d p d x. So, this now we integrate. So, d 2 u d y 2 is equal to 1 by mu d p d x. And we know that d p d x is a constant. So, now, if we integrate it d u d y is equal to 1 by mu d p d x into y plus c 1. And u is equal to 1 by mu d p d x y square by 2 plus c 1 y plus c 2. How do you get c 1 and c 2? We use the 2 boundary conditions boundary condition 1 at y is equal to 0 u is equal to 0. This is the no slip boundary condition. So, you have c 2 equal to 0, boundary condition 2 at y equal to h u equal to u 1.

So, you have u 1 is equal to 1 by mu d p d x, h square by 2 plus c 1 h; that means, c 1 is equal to u 1, by h plus 1 sorry minus this is minus 1 by mu d p d x into h by 2. So, u is equal to 1 by mu d p d x, 1 by 2 mu d p d x, into y square then c 1 y minus y h plus u 1 y by h. So, you can clearly see that this satisfies the boundary condition at y equal to 0 u is 0 and at y equal to h also u is 0. So, this is the velocity profile and the very interesting thing you can observe, in this velocity profile, you have one part which is purely because of the pressure gradient. And another part this is purely because of the shear. And the result end is just the algebraic some of this 2.

Why it is so? It is so because this equation is a linear equation. So, if u equal to some u pressure driven in the solution and u equal to u shear driven in the solution. Then u equal to u pressure driven plus u shear driven is also a solution of this equation. Now what is ask from ask is not u, but what is q, or if Q is 0 then what is u 1, u 1 is not given. So, late us calculate what is q. So, Q is equal to the what is Q integral u d y this is Q per unit with from y equal to 0 to y equal to h.

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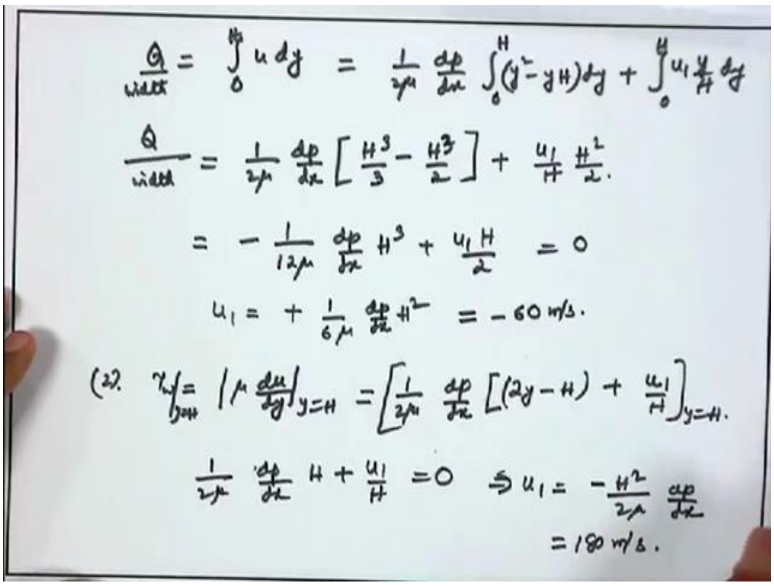

So, 1 by 2 mu d p d x, plus u 1 y by h d y from 0 to h; this is Q by width. So, Q by width is equal to 1 by 2 mu, d p d x, to h cube by 3 minus h square by 2, sorry, h cube by 2 right plus u 1 by h into h square by 2. So, now, this is minus this is 3 into 2 is 6 into 2 is 12. So, 1 by 12 mu d p d x into h q, plus u 1 h by 2 and this is equal to 0.

So, what is u 1? U 1 is minus. So, sorry plus 1 by 6 mu d p d x into h square. 1 by 6 mu d p d x into h square. So, if you put the values of this then this, will come out to be minus 60 meter per second. The second part of the problem is what should be u, so that tau all equal to 0. Before that how is it possible that you are having a pressure gradient, but still the flow rate is 0. See physically what is happening in this problem is, that there is a driving pressure gradient which is minus 900 Pascal meter. So, that is trying to drive the flow in the positive direction.

If u 1 also drives the flow in the same direction that there will be a net flow, but in this physical problem we are seen that u 1 is coming out to be minus 60 meter per second; that means, u 1 is in the opposite direction. So, there is a flow rate that is being attempted to be created by the pressure gradient in the forward direction, and the shear in the negative direction. And some total is coming out to be 0 for this u is equal to minus 60 meter per second.

Now, wall shear stress. Wall shear stress as we have discussed is mu d u d y and our condition was given that at y equal to h this will be 0. So, what should be u 1 such that at y equal to h tau all is 0. So, mu d u d y, tau all at y is equal to h is mu d u d y at y is equal to h. So, u is given. So, what is d u d y at y equal to h? 1 by 2 mu d p d x into 2 y minus h, plus u 1 by h, at y is equal to h. So, 1 by 2 mu, d p d x into h, plus u 1 by h, this is equal to 0. So, u 1 is equal to minus h square by 2 mu d p d x.

The y this is 0, because it is given that the wall shear stress at y equal to h is 0. So, this is this u 1 is 180 meter per second. Now we have worked out few problems. The next case that we will consider is the special case which is called as thin film flow. Thin film flow under fully developed condition.

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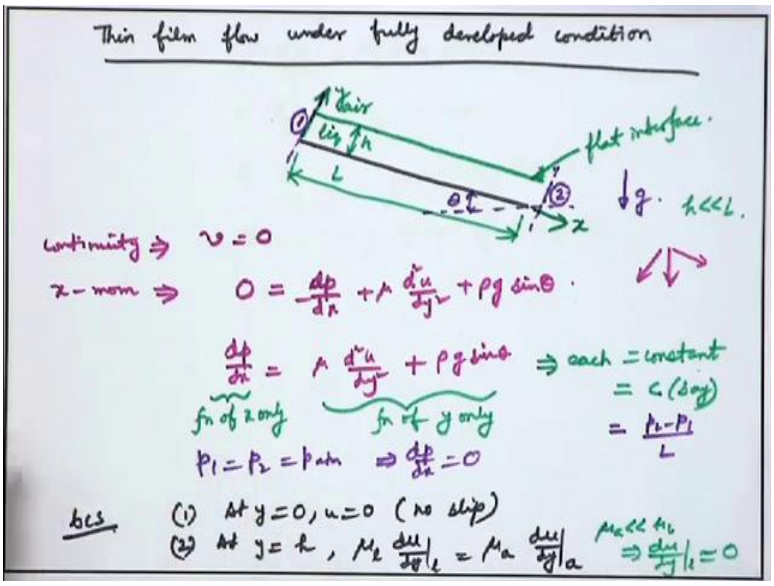

Let us say, that there is an inclined plate and the plate is incline at an angle theta with the horizontal, whereas, the gravity is acting vertically downward fashion. We set up the x and y axis such that x axis is along the inclined plane, and y axis is perpendicular to that. And there is a thin liquid film with a flat interface. If it is a curved interface it is a significantly more complicated problem that we will not take up in this particular lecture.

So, this is a flat interface. There is a liquid and there is air and this film thickness h is much less than the length l of the incline. So, now, let us write for that for fully developed state, the for fully developed flow v is equal to 0, x momentum equation. So, this is the consequence of continuity, x momentum equation left hand side is 0, minus d p d x plus mu d 2 u d y 2, and there will be an additional body force which is, if this is g along the incline it is g sin theta and here it is g cos theta. So, plus rho g sin theta; fundamentally there is a problem with this equation. And we need to consider the y momentum.

First of all, p here is a function of both x and y all right, but if the variation if the height h of the domain is much less than l, then the pressure variation within this thin film is much less than the pressure variation along it. And therefore, in that case p is not a function of y. Otherwise if this film is a thick film then p can be a significant function of y. So, p is not a function of y p is a function of x only and u is a function of y only. So, you can write d p d x is equal to mu d 2 u d y 2 minus rho g sin theta. So, this d p d x, this is the function of x only. And this is the function of y only. This is the constant.

So, it can be absorbed in either function of x or function of y. So, this is plus right. Now function of x is equal to a function of y, if h is equal to a constant. So, we can say that implies each is equal to constant. So, if you say that it is a constant, and then we give this 2 sections name as this 2 and this is 1, then this d p d x is nothing, but p 2 minus p 1 by l. Now here both p 1 and p 2 are p atmosphere. This means d p d x is equal to 0.

So, here the film is falling totally by the effect of gravity. So, you have mu d 2 u d y 2, I am writing in the in a new page.

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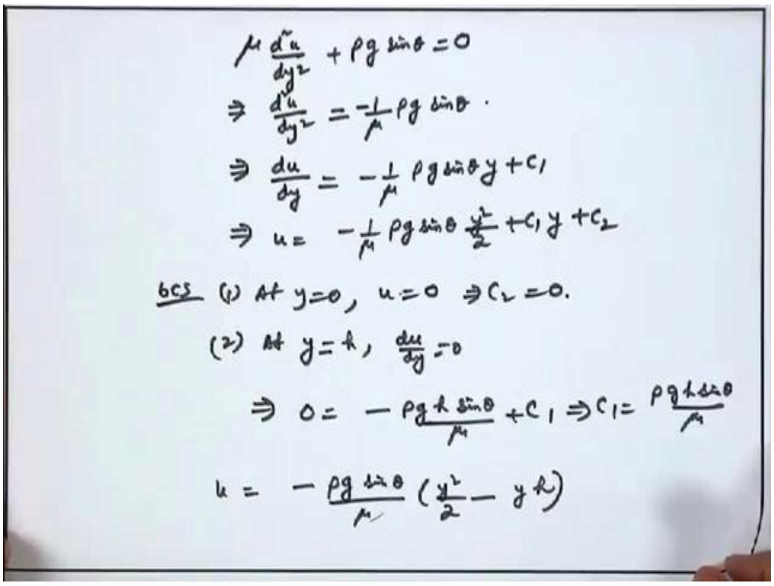

This means d 2 u d y 2 is equal to 1 by mu minus 1 by mu, rho g sin theta. So, if you integrated twice, then d u d y is equal to minus 1 by mu rho g sin theta y, plus c 1 and u is equal to minus 1 by mu rho g sin theta y square by 2 plus c 1 y plus c 2. Now what are the boundary conditions. Let us come to this figure once again. Let us write the boundary conditions here.

First boundary condition is pretty clear. At y is equal to 0 u is equal to 0. This is no slip. Then at y is equal to h, what is the boundary condition? See at y equal to h, there should be a continuity in shear stress. So, you can write that mu liquid d u d y at the liquid is equal to mu of air into d u d y air. Now because mu air is much less then mu liquid, we can say that d u d y in the liquid side which is our domain is roughly equal to 0. So, very often students use this boundary condition without understanding where from it comes. It comes from nothing, but the continuity of shear stress across the boundary across the flat interface.

So, when the viscosity of here is much less than the viscosity of liquid then that boils down to the d u d y in the liquid must be very small or approximately 0. So, if that with the case let us note down the boundary conditions here again. Boundary conditions number 1, at y is equal to 0 u is equal to 0; that means, c 2 is equal to 0. Number 2, at y equal to h d u d y equal to 0; that means, 0 is equal to minus rho g h sin theta by mu plus c 1. So, c 1 is equal to rho g h sin theta by mu.

So, u is equal to minus rho g sin theta by mu, into y square by 2 minus y h. This is the velocity profile. So, you can clearly see that the velocity is driven by this rho g sin theta. That is the gravity effect. This is typical gravity driven flow. We will consider another example where now, So, for we have considered the use of Cartesian coordinate system, but another example where cylindrical coordinate system is important. And that is an example which refers to one of the very important applications in engineering that is pipe flow.

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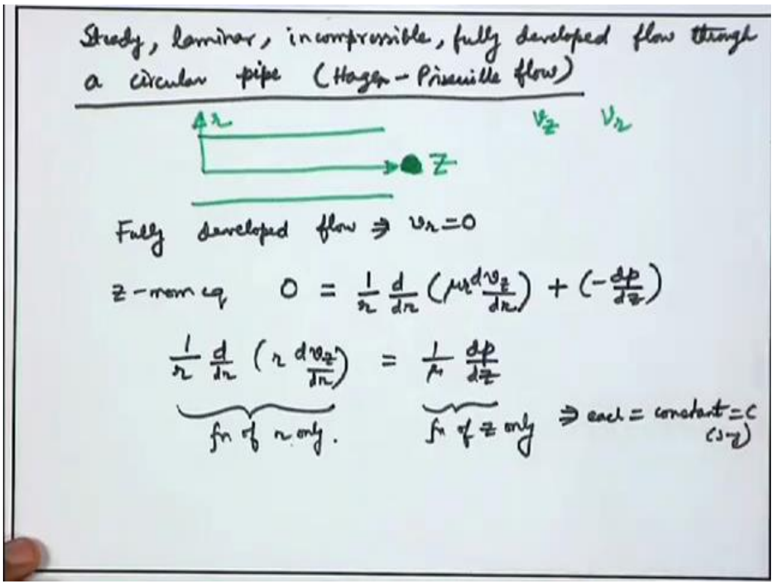

So, we will consider study laminar incompressible fully developed flow through circular pipe. This is called as Hagen poiseuille flow. So, I will just talk about the only settle changes from the rectangular to the Cartesian system. See for the pipe the convenient coordinate system is the r theta z.

This is z this is r and u have the theta, but because of the theta symmetry or the azimuthal symmetry instead of x y system it is z r system. So, u will be replaced with v z, and y and v will be replaced with v r. So, for fully developed flow you have v r is equal to 0. And the z momentum equation left hand side will be 0 for fully developed flow, right hand side this is the only change. Because of the cylindrical coordinate system, the d 2 u d y 2, will become 1 by r d d r, of mu r, d v z b r.

So, this is the only change in the form of the derivative because of change of coordinate system from Cartesian to cylindrical, plus minus d p d z that will remain as it is. So, for mu equal to constant, 1 by mu sorry 1 by r, d d r of r, d v z d r is equal to 1 by mu d p d z. So, this is the function of r only. This is the function of z only. This implies that each is equal to constant, equal to c say. So, 1 by r, d d r, of r d v z d r, is equal to c. So, if you integrate it r, d v z d r, is equal to c r square by 2 plus c 1.

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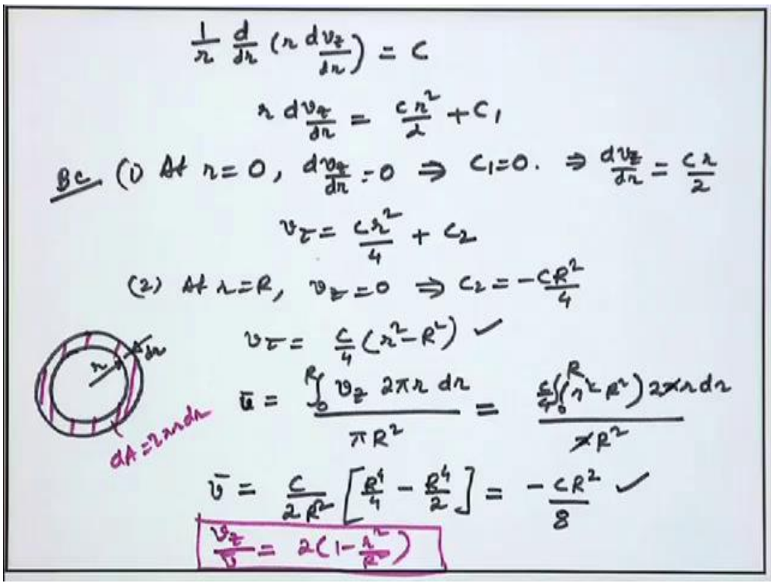

Now, you can amply the boundary condition, that at the central line v z is maximum, that is at small r equal to 0, d v z d r equal to 0; that means, c 1 equal to 0. So; that means, you have d v z d r is equal to c r by 2. So, if you integrate it, then v z is equal to c r square by 4, plus c 2. Now let us assume here that small r is equal to capital R is the radius of the pipe. This is the central line. So, the second boundary condition at small r is equal to capital R v z is equal to 0; that means, c 2 is equal to minus c r square by 4.

So, you have v z is equal to c by 4, into small r square minus capital R square. Now what is the average velocity v average? This is integral of v z into a elemental area 2 pi r d r, by pi r square. This is the total volumetric flow rate elemental area. So, this this is the circular section. If you consider at a radial location r, at thin street of width d r, then this d a is 2 pi r d r. So, let us integrate this, v z into 2 pi r d r. So, v z is c by 4 into r square minus r square into 2 pi r d r, by pi r square 0 to r.

So, v average is this pi pi gets cancelled. So, c by 2 r square then integral of r cube d r is r 4 by 4, minus integral of r d r is r square by 2. So, that becomes r 4 by 2. So, this becomes or does it become it becomes minus c R square by 8 right. So, if you substitute this value of c here, then v z by v average is equal to 2 into 1 minus small r square by capital R square. This is the fully developed flow through circular pipe the velocity profile. You can see that it is very similar to the parallel plate channel case, only thing that it is expressed in terms of a different coordinate system. Now for practical engineering consideration what is more important for ask as we have discussed is not just the velocity profile, but what is the pressure drop.

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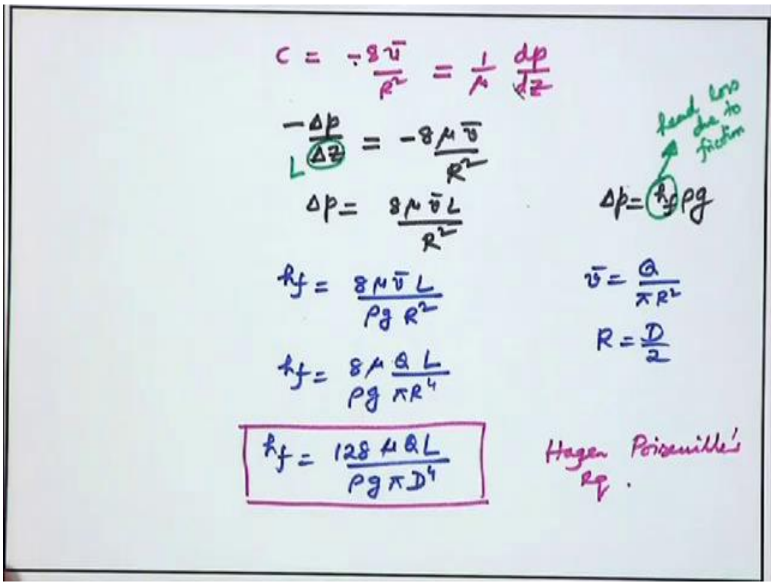

So, the pressure drop c. So, you can write from this this slide, that this c is equal to minus 8 b average by r square, and c is nothing, but 1 by mu d p d z. So, this you can write delta p by delta z is equal to 8 minus 8 mu v average by r square. If we write this delta p, now can you tell that this d p d z will it be positive or negative for a positive v average, for a positive v average d p because viscosity is positive and here you have negative sin. So, positive v average will mean that you have negative d p d z. So, let us call that negative d p d z as minus dealt p by delta z.

So, delta p and let us say this data z is l the length of the pipe or the length over which you are measuring the delta p, is 8 mu v average l by r square. Now you can write delta p in terms of length unit by calling h f rho g. So, why do we call delta p as a h f rho g, see this delta p why are you requiring a pumping power? In the pipe you would have required no pumping power, had they are been no viscous resistance, but here you are trying to apply a pressure gradient through a pumping power; so that you can maintain the flow even in the presence of viscous resistance.

So, delta p you can write h f rho g, where h f is an head loss due to friction. Head is energy per unit weight in hydraulics head is known as head is equivalent to energy per unit weight. So, h f is equal to 8 mu v bar l by rho g r square. And v is nothing, but Q by pi r square. So, h f is equal to 8 mu Q by pi r square into r square is r to the power 4 by rho g l. And r is nothing, but often in engineering you know instead of rad diameter instead of radius we expressed it in terms of diameter r is equal to d by 2. So, h f is equal to 128 mu Q l by rho g pi d to the power 4.

This is known as Hagen poiseuillies equation. So, you require you see that the head loss is inversely proportional to the 4th power of the hydraulic diameter or the diameter of the pipe, is proportional to Q proportional to l. So, all this thing are linearly proportional mu Q l, but d is inversely related to 4th power of d. So, if you make d smaller and smaller and smaller h f will be larger and larger and larger. So, to maintain the same Q, you require huge pumping power to drive the flow through a very narrow channel and that is one of the great challenges in driving flow of fluids through micro channels and Nano channels, which is itself a very interesting topic.

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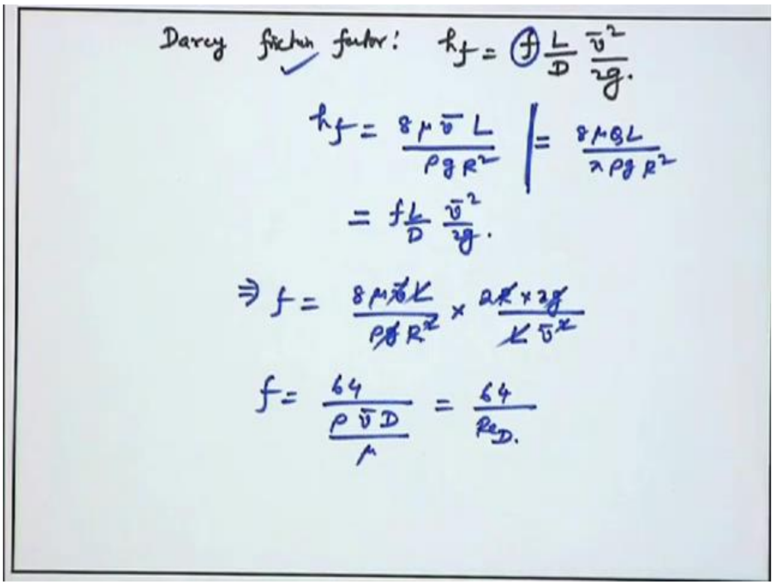

Now, we have introduced one friction factor c f, I will introduce another friction factor f which is called as Darcy friction factor, h f is defined as f l by d v average square by 2 g. This f is called as Darcy friction factor this is just a non-dimensional pressure drop the fanning friction factor is a non-dimensional walls shear stress, because in a fully developed flow the shear force is balanced by the pressure force. So, this a fan c f are related with each other, but let us find out f. So, h f is 8 mu v average I am just writing one of the equations in their previous slide, by rho g r square and v average is Q by pi r square.

So, 8 mu Q l by pi rho g r square; living this a part we are not going to use this. So, this h f is f l by d v average square by 2 g. So, f is equal to 8 mu v average l by rho g r square, into in place of d we will write 2 r into 2 g by l v average square. So, now, you can see that g gets canceled, one v gets canceled, l gets canceled, and one r also gets canceled. So, f is equal to 8 into 2 into 2 that is 32 and r becomes d by 2. So, 32 into 2, 64 by rho v average d by mu. So, f is 64 by Reynolds number. So, for fully developed flow through for study laminar incompressible fully developed flow through a circular pipe, the friction factor versus Reynolds number is given by f equal to 64 by Reynolds number.