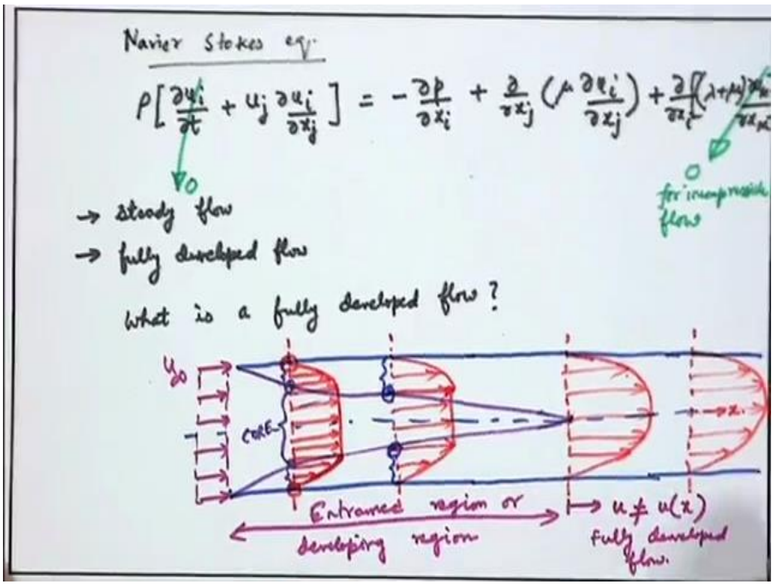
Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 53
Navier-Stokes equation-Part-III

We have discussed in our previous lecture, the governing equation for viscous flow for Newtonian, Stokesian, homogeneous and isotropic fluid and the corresponding equation known as Navier stokes equation, let me write that once more.

(Refer Slide Time: 00:44)



So, navier stokes equation. So, I am writing the navier stokes equation for the general case, with this equation it is important that we try to understand what are the challenges in solving this equation and that we have already discussed in the previous session. Now let us come to some special case.

So, first of all, we consider the case when the flow is incompressible if the flow is incompressible this term is clearly 0. So, this is 0 for incompressible flow. So, now, let us try to study a special case when the left hand side of the navier stokes equation becomes 0 and that special case, we consider as steady flow. So, for steady flow, this term is 0 and the next concept that we are going to study is fully developed flow. So, let us try to understand, what is a fully developed flow? Let us say that we have a channel made of 2 parallel plates, this is the simplified version of for rectangular channel where there are other boundaries, but for simplicity let us considered channel with 2 parallel plates.

Now, let us say that fluid comes from first stream with velocity u infinity, now this fluid flow will be resisted as the fluid encounters the solid boundary. So, when the fluid encounters the solid boundary what happens let us try to draw the velocity profile at a cross section. So, first this velocity is uniform. Next let us consider another section, let us say this pipe is quite long pipe or channel so that we can get a fear understanding of what happens as we proceed along the axial direction of the pipe. Axial direction of the pipe is x now if we draw a velocity profiles here, what happens at the wall? The velocity is 0 because of the no slip boundary condition.

Now, further away from the wall the velocity increases. So, you will get a velocity profile that has being developed till you come to a location. So, I am drawing the velocity profile, till you come to a location where the velocity no more velocity gradient exist. So, this region within which the velocity profile develops and this velocity profile develops because of viscous effects. This region is called as boundary layer we are not going to discuss about what is boundary layer in this particular course because it is not within the scope of this course, but physically it is a viscous layer adjacent to a solid boundary within which the effect of the momentum disturbance because of the presence of the boundaries failed.

So, the momentum with these disturbance created by the solid boundaries such that the fluid comes to 0 velocity when it is in contact with the solid boundary and then what happens is that the velocity increases as we go away from the solid boundary then the velocity is uniform and after this again the velocity decreases like this (Refer Time: 08:11). Then let us consider another section if we consider another section now before going to another section let me discuss something which is interesting the velocity in this way the velocity variation in these way divided into 2 regions, 1 region up to this we call as core region sorry boundary layer region and beyond this, this is called as core region.

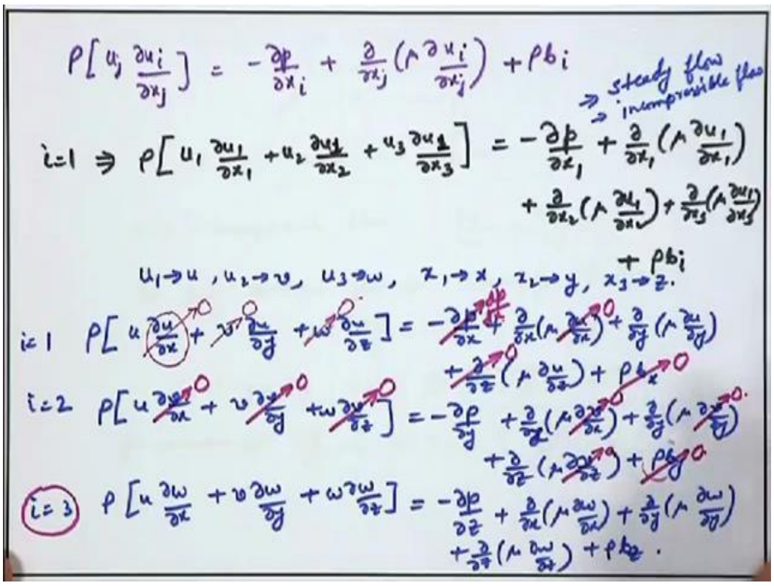
So, boundary layer region is adjacent to the solid boundary and outside the solid boundary it is called as the core region, then let us go to another section which is further away from the solid boundary further away from the entrance and let us try to draw the velocity profile. So, we will see that because this fluid is now more and more into the channel more and more fluid will be influenced by the effect of the wall. So, the boundary layer will be thicker. Let us say that the boundary layer is now up to this much. So, this local boundary layer thickness is up to this. So, what happens outside the boundary layer? See the flow rate, it is conserved.

So, the slowing down effect within the in the near wall region is compensated by the acceleration of the fluid along the core region. So, that you know that there net flow rate across each section is conserved. So, now, if we draw the locus of all this points all this points which are in the edge of the boundary layer we can draw an imaginary line like this and this imaginary line because it is symmetric on the 2 sides of the channel this imaginary line will converts to a point on the central line and this is a very special point. If you consider, if you considered the flow then you will realize that beyond this what happens the entire fluid fields the effect of the solid boundary before this only the fluid within the boundary layer fields explicitly the effect of the solid boundary, but when you reach up to here you will find that the entire fluid has understood that there is a solid boundary.

It does not happen immediately, I am not going into the details, I mean actually there is some sort of a region where there is a gradual transition from this behavior to this behavior, but let us assume that it becomes fully developed that is the velocity profile becomes of this shape which does not change with the axial direction anymore. So, if you draw the axial coordinate the velocity does not vary with axial coordinate anymore. So, we can write that from here onwards u that is the axial component of velocity is not a function of the axial coordinate x. So, this is called as fully developed flow and this region is called as entrance region or developing region.

So, now how do the equations get simplified for a fully developed flow that is the next question? And that is the important question that we are looking for in today’s lecture.

(Refer Slide Time: 14:29)



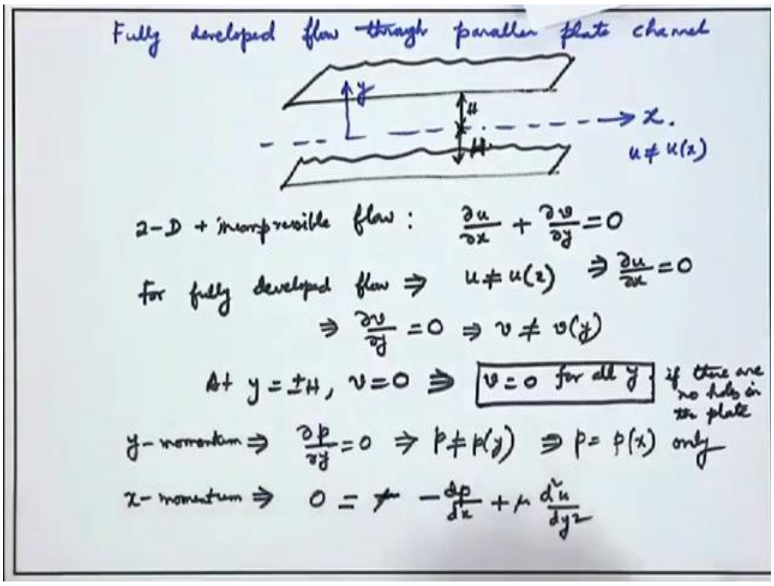
So, let us write that for steady flow the navier stokes equation rho u j del u y del x j assuming incompressible flow the last term in the right hand side is 0. So, let us write the component with i equal to 1, i equal to 1 will mean rho u 1 del u 1 del x 1 plus u 2 del u 2 del x 2 plus u 3 del u 3 del x 3 sorry, u 3 del u 1. All are 1 u 3 del u 1 del x 3 is equal to minus del p del x 1 plus del, del x 1 of mu del u 1 del x 1 plus del, del x 2 of mu del u 2 del x 2 mu del u 1 del x 2 plus del, del x 3 of mu del u 1 del x 3.

If there is a body force we will add the body force. So, let us write as rho b i which we did not write in the previous page if you want you just add rho b I for the body force, so plus rho b i. Now let us try to look write it in the alternative notation in which books will commonly write in the under graduate level. So, u 1 is u, u 2 is v u 3 is w x 1 is x, x 2 is y and x 3 is z. So, you have rho u del u del x. So, this equation remember; we are writing for steady and incompressible flow. So, already we have made to assumption steady flow incompressible flow.

So, u this equation will become u del u del x plus v del u del y plus w del u del z is equal to minus del p del x plus del del x of mu del u del x plus del del y of mu del u del y plus del del z of mu del u del z plus rho b x. So, this is I equal to 1, similarly I equal to 2 just replace u with v. So, rho u del v del x plus v del v del y plus w del v del z is equal to minus del p del y plus del del y of mu sorry; del del x of mu del v del x plus del del y of mu del v del y plus del del z of mu del v del z plus rho b y and I equal to 3 is rho u del w del x plus v del w del y plus w del w del z is equal to minus del p del z plus del del x of mu del w del x plus del del y of mu del w del y plus del del z of mu del w del z plus rho b z.

So, this 3 are the x y and z components of the navier stokes equation that is what is commonly between in terms of the standard notions used for under graduate level text books now we will see that how this equations are influenced for fully developed flow. So, we will take an example of fully developed flow through a parallel plate channel.

(Refer Slide Time: 20:21)



Fully developed flow through parallel plate channel. So, let us consider yes let us consider 2 parallel plate channels sorry; 2 parallel plates making a parallel plate channel and let us consider that the central line coordinate is the x and cross coordinate is y.

So, now we can write for fully developed flow. So, see the entire channel will not have fully developed flow. So, if you are interested to consider fully developed flow we will consider beyond this regime. So, we are not considering what is happening at the entrance region of the pipe or the channel. So, now, for fully developed flow we are writing u is not a function of x now another interesting thing is; what is a parallel plate channel? So, a parallel plate channel is essentially made of 2 parallel plates that are infinitely wide. So, if they are infinitely wide then the z directions are of not great importance what happens along the z direction. So, essentially it is a 2 dimensional flow.

So, for 2 dimensional incompressible flow you have del u del x plus del v del y equal to 0.

Now, for fully developed flow u is not a function of x; that means, del u del x equal to 0. So, if del u del x is equal to 0 then from this equation what we have del v del y is equal to 0; that means, v is not a function of y. So, v is not a function of y then what is the value of v. So, we know let us say that H is the half height of the channel. So, if we know the value of v at some y that same value of v should be there for all values of y. So, we know that at y is equal to plus or minus H v is equal to 0 from this what we can conclude I ask this question in the chat we note from the boundary condition that y equal to plus minus H because of no penetration boundary condition v is 0.

So, then what can we conclude out of this please answer in the chat I am not getting an answer in the chat, perhaps today's session is the afternoon session. So, everybody had a good very lunch and the response is an indicative of how the good the lunch was. So, let me try to answer this at y equal to plus minus H v equal to 0. So, what it means is that see v is not a function of y. So, at y equal to plus minus H v is equal to 0 means at all values of y v must be 0 because v is not a function of y. So, v is equal to 0 for all y. So, v is identically equal to 0 for all y.

Can you tell can you give me an example when v will not be equal to 0 again I am asking a question in the chat can you tell or can you give an example of a situation when even if the flow is fully developed v is not equal to 0 even if the flow is fully developed please answer in the chat. Let us say that there are holes in the plate if there are holes in the plate then what will happened then the fluid can penetrate the solid boundary because there are holes in the solid boundary in that case you will have a flow through the plate so; that means, v is equal to 0 for all y, if there are no holes in the plate no holes in the plate holes or force whatever.

Now, let us write the x momentum equation or let me bring back the slide. So, it will help. So, what we have assumed till now? Steady flow incompressible flow fully developed flow and 2 dimensional flow. So, let us now simplify first of all because it is a 2 dimensional flow this equation is not relevant this is the third direction. So, the 2 directions x and y they are important. So, just now we have seen that the flow is fully developed; that means, v is equal to 0 if there are no holes on the plate well there is a body force along y, but that will not give rise to a significant variation of velocity because the channel height is not significantly large that we have assumed.

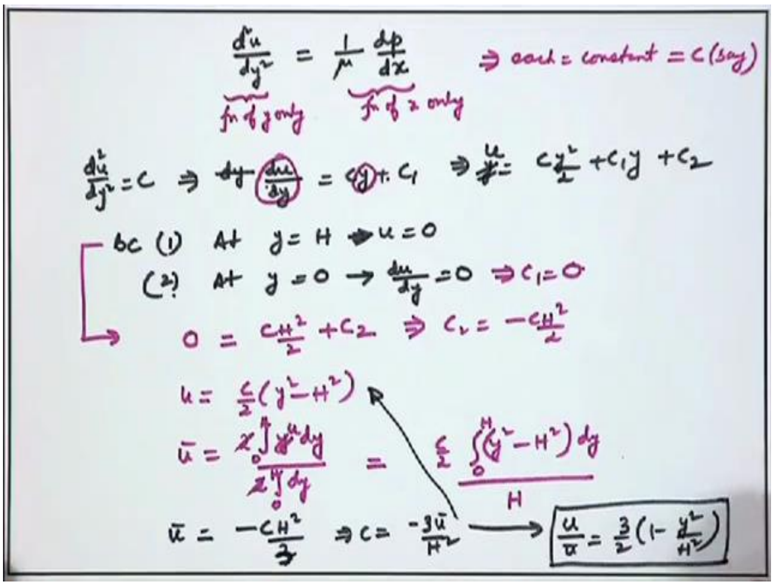
If the channel height is significantly large the body force along y may play its some role. So, we are not considering any body force along y we are also we are considering the y momentum equation first and you will soon understand why. So, this equation is 0 this equation is 0 this equation this is 0 all these are 0; that means, we can write del p del y equal to 0. So, y momentum equation this means p is not a function of y; that means, p is a function of x only. So, p is a function of x only we will use this in the x momentum equation. Let us assume that it is a horizontal pipe or a horizontal channel. So, p is a function of x only means this is d p d x. It is a horizontal channel means b x is equal to 0.

If it is an incline channel then this will have a rho g component and then this rho g component may be coupled with d p d x. So, it includes d d x of p plus rho g into H. So, that is called as piezometric pressure p plus rho g H. So, instead of d p d x, we can write d p star d x where p star includes the gravity effect also, but here for simplicity we are not considering that and we are considering that it is just a horizontal channel now x component of velocity del u del x. So, when the flow is fully developed del u del x is equal to 0 when the flow is fully developed and there are no wholes on the solid boundary v is also 0 and because 2 dimensional flow w is also 0 because u is not a function of x this is 0 and there is no z variation because it is a 2 dimensional flow. So, this term is also 0.

So, you are left with see now the left hand side of the navier stokes equation has become 0 identically equal to 0 the left hand side of the navier stokes equation has identically become 0 and this is called as stokes equation the difference between the navier stokes equation and the stokes equation is that the navier stokes equation is non-linear, but the stokes equation because of the elimination of the non-linear terms it becomes linear. So, let us write the x momentum equation for the for this case 0 is equal to now we also assume that mu is constant or it because it is hum homogeneous fluid you can take mu out of the derivative. So, mu and also see u is a function of y only u is no more a function of x. So, this partial derivative becomes ordinary derivative you can write d d y instead of del del y.

So, mu sorry, first d p d x term is there minus d p d x plus mu d 2 u d y 2. So, for fully developed flow through a parallel plate channel we can write d 2 u d y 2 is equal to 1 by mu d p d x.

(Refer Slide Time: 34:15)



Next what we can tell about this, this is a function of y only, this is a function of x only. So, this is the function of y only and this is the function of x only and they are equal c f general function of y can become equal to a general function of x only if H is equal to a constant let us say that constant is c. So, now, you have d 2 u d y 2 is equal to c if you integrate it you have d u sorry, d u d y is equal to c y plus c 1 and y is equal to c y square by 2 plus c 1 y plus c 2 where c 1 and c 2 are 2 arbitrary independent constants of integration.

Now, what are the boundary conditions see because of symmetry of the problem with respect to the center line we can as well solve for this shaded domain half of the domain. So, at y equal to H 1 boundary condition is pretty clear that at y equal to H u is equal to 0. In fact, y equal to plus minus H because our domain is only the shaded domain it is just y equal to H u equal to 0 second boundary condition what is the boundary condition at y equal to 0 at y equal to 0 that is the central line the velocity is maximum. So, d u d y equal to 0 this is also following from the central line symmetry.

So, at y equal to 0 d u d y equal to 0; that means, c 1 equal to 0 from this equation then at y equal to H u equal to 0. So, sorry this will be u see; that means c 2 is equal to minus c

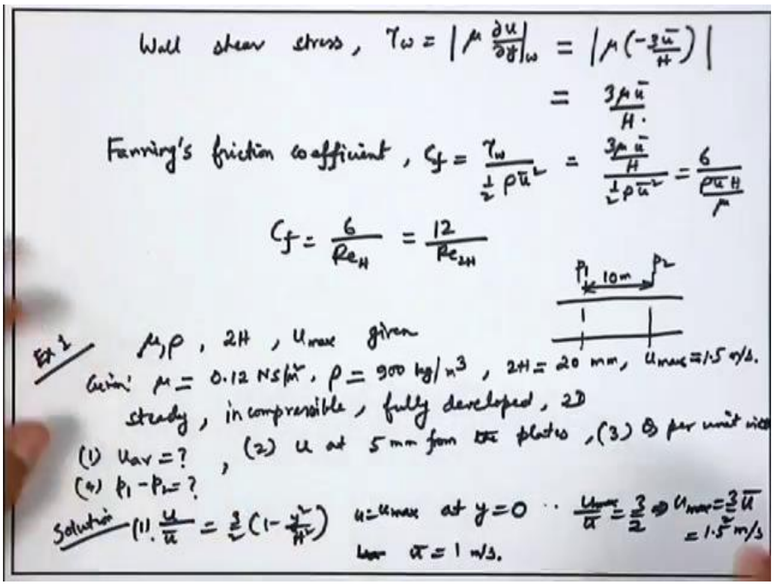
H square by 2. So, the velocity profile is u is equal to c by 2 into y square minus H square see because you varies quadratically with y this is a parabolic velocity profile. So, the fully developed velocity profile that we have drawn this is parabolic in shape now the whole point is that all though we have mathematically determined the expression for u as a function of y, but we have to remember that c is not a parameter that has been determined.

So, we have to express c in terms of other parameters what are the other parameters let us say we want to express the c in terms of the average velocity what is the average velocity average velocity is integral u d y by integral d y. So, what is this integral u d y we will represent per unit with the volume flow rate. So, had the same volume flow rate been there, but the velocity would have been uniform then that equivalent uniform velocity is called the average velocity. So, the average velocity is an equivalent uniform velocity which maintains the same volume flow rate as that of the actual case. So, now, let us put the limits of integration.

So, integral u d y is c by 2 integral y square minus H square into d y and integral d y from 0 to H is H. So, u average this is H cube by 3 this is H cube, minus 2 H cube by 3. So, minus c H square by 2 right, 3 not 2 minus c H square by 3 so; that means, c is equal to minus 3 u average by H square. So, if you substitute this here then what we get is u by u average is equal to 3 by 2 into 1 minus y square by H square this is the velocity profile now as the engineers we are of course, interested about the velocity profile, but the practical interest is that how much pumping power is required to drive the flow. So, what is the pumping power required to drive the flow to understand that first of all we need to find out what is the wall shear stress of course, the pumping power is required with pressure drop and we will learn that carefully when we consider another example of flow through a circular pipe, but let us at least conclude these exercise here by finding out the wall shear stress.

The wall shear stress tau wall is this much mu del u del y at the wall. So, remember that it is actually mu into del u del y plus del v del x, but del v del x is 0 at the wall. So, mu del u del y at the wall. So, now, u by u average is 3 by 2 into 1 minus y square by H square. So, these becomes mu del u del y is. So, if you differentiate this expression this will be 6 minus 6 y by 2, so minus 3 y by H square, so at y equal to H minus 3 by H, so minus 3 u average by H. So, now, it is customary to express this tau average. So, fast of all let me write this 3 mu u average by H. So, the Fanning's friction coefficient c f is equal to tau all by half rho u average square.

(Refer Slide Time: 42:20)



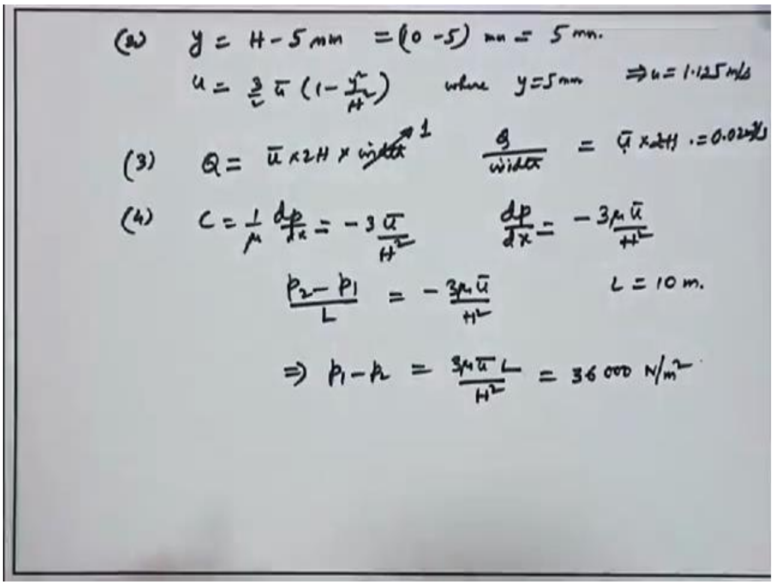
So, half rho u average square as a unit of stress or pressure. So, it is customary to non dimensionalize tau wall with respect to that so 3 mu u average by H by half rho u average square, so, 6 by rho u average H by mu, so rho into u average into length by mu is nothing, but the Reynolds number based on H. So, c f is equal to 6 by Reynolds number based on H instead of H if you use 2 H then it is 12 divided by Reynolds number based on 2 h, 2 H is the height of the channel total height H is the half height. So, with this introduction of fully developed flow between 2 parallel plates let me see whether we can work out an example.

So, if you go through page number four of your lecture notes then in page number four of the lecture notes there is a problem it is purely numerical problem. So, I am not going to tell you about the numbers present in the problem because you can refer to your notes and see, but I am just schematically trying to represent the problem and give you the concept. So, example 1, oil of viscosity say mu and density rho flows between 2 large parallel plates which are kept at a distance 2 H equal to 20 millimeter the maximum velocity is 1.5 these are given u max these are given 1.5 meter per second the flow is steady incompressible and fully developed.

So, what is given mu is 0.12 Newton's second per meter square rho is nine hundred kg per meter cube 2 H is equal to 20 millimeter and instead of u average. Now it is u max that is given that is 1.5 meter per second flow is steady incompressible fully developed and because it is a parallel plate channel it is 2 dimensional. So, you have to find out what is u average what is the velocity at 5 millimeter from the plates the discharge per unit width q per unit width and the difference in pressure between 2 points 10 meter apart. So, if you draw the pipe or the channel this is p 1, this is p 2, this is 10 meter, p 1 minus p 2 is equal to what?

Let us work out this problem. So, solution first part what is u average? So, u average, you have to remember that u by u average is 3 by 2 into 1 minus y square by H square. So, given u max u max is at what why u equal to u max at y equal to 0 that is the centre line. So, at y equal to 0 u by u average is 3 by 2. So, u max is 3 by 2 u average. So, u max is 1.5 meter per second, hence u average is 1 meter per second, this is the first part.

(Refer Slide Time: 50:24)



Second part u equal to 5 millimeter from the plates 5 millimeter from the plates is y is equal to H minus 5 because y is measured from the central line. So, 5 millimeter from the plates means H minus 5 millimeter. So, H is 10 millimeter.

So, that is equal to 5 millimeter. So, in the velocity profile u is equal to 3 by 2 u average into 1 minus y square by H square where y equal to 5 millimeter so; that means, u becomes 1.125 meter per second. This is the final answer what is the volume flow rate q is u average into 2 H into width for unit width or per unit width q by width is equal to u average in 2 H. So, the values are given values of u average and H are given. So, you can calculate this as 0.02 meter cube per second this is 10 millimeter and this is 1 meter per second.

Finally the pressure p 1 minus p 2, now, if you see there was an expression c is equal to minus 3 u average by H square and this is nothing, but 1 by mu d p d x, c is 1 by mu d p d x. So, let me write in this page; c is equal to 1 by mu d p d x is equal to minus 3 u average by H square. So, what is d p d x? Minus 3 mu u average by H square because d p d x is a constant, p versus x is linear. So, d p d x is p 2 minus p 1 by L is equal to minus 3 mu u average by H square where L is this 10 meter that is given. So, p 1 minus p 2 is 3 mu u average L by H square. This is 3; 36,000 Newton per meter square.