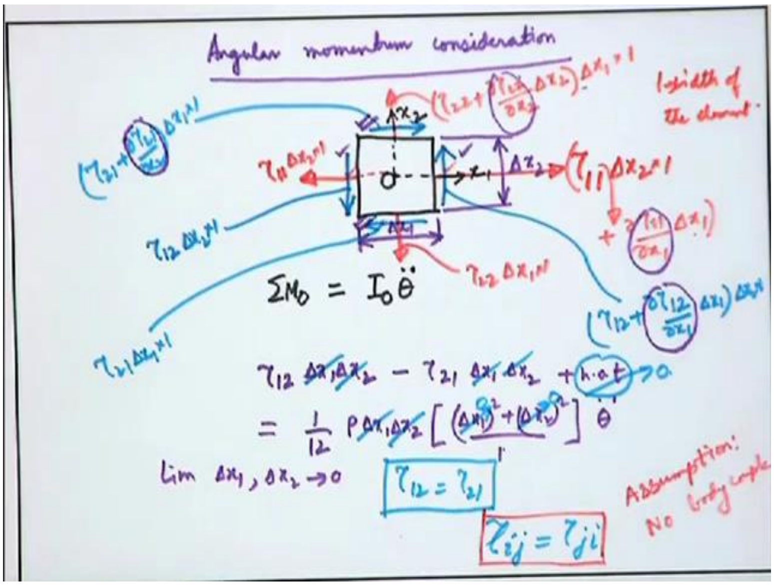
Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 52
Navier - stokes equation- Part-II

Now so far we have considered only linear momentum conservation. So, we will take little bit of distraction from that, and we will now think of angular momentum consideration. The linear momentum consideration is very common; angular momentum consideration in fluid mechanics is not always so obvious but we need to carefully look into it.

(Refer Slide Time: 00:53)



So, angular momentum consideration; so let us take a now angular momentum just like linear momentum refers to translation, angular momentum refers to rotation. Now let us say that we are interested about rotation with respect to z axis, now we can decouple the rotation in terms of rotation with respect to x axis, rotation with respect to y axis and z axis; x means x 1, y means x 2 and z means x 3. So, we can isolate these effects and consider 1 at a time.

So, we can consider for example, rotation with respect to the z axis or x 3 axis, then it is important that we consider only the forces which take place which are there in the x y plane or x 1, x 2 plane. So, let us say this is the x 1 axis and this is the x 2 axis; and let us say that we give dimensions to this such that this is delta x 1 and delta x 2. Now we are interested to have rotation with respect to or an equation for rotation with respect to x 3 axis an axis that passes through O. So, for rotational motion we can write resultant moment of all forces with respect to an axis which is normal to this plane and passing through O is equal to the moment of inertia with respect to the same axis times the angular acceleration ok.

So, now resultant moment of forces. So, let us identify the forces here, let us write the normal forces and the tangential forces. So, this force this is tau what is the first index? the first index is 1 because the direction normal is 1 the second index is also 1. So, tau 1 1 into delta x 2 into 1, let us say 1 is the width of the element. If this is tau 1 1 into delta x 2, essentially, just a matter of nomenclature, if we call this as tau 1 1, this will be tau 1 1, plus del tau 1 1 del x 1 into delta x 1; because the difference in x coordinate between the 2 phases is delta x 1. Interestingly we will see that this increment or this higher order effect will not matter for calculating the moments, this will matter for calculation of forces, but not calculation of moments. These are the normal forces; so let us write the normal forces also along this. So, this is tau 2 2 into delta x 1 in to 1, this is tau 2 2 plus del tau 2 2 del x 2, into delta x 2, into delta x 1 into 1. These are the normal forces now this forces I have written just for completeness, but this forces actually do not contribute to any moment, why because all this forces pass through this point O therefore, this do not contribute to the moment with respect to an axis passing through O.

Then let us draw the tangential forces. So, first let us show their proper directions and then we will write the force. So, I am showing the forces in the proper direction as per the sign convention we discussed earlier, and then I am writing the description of the forces. So, this force what is this? This is tau 2 1 sorry tau 1 2 direction normal is 1. So, tau 1 2 into what is the area delta x 2 into 1, what is this? This is tau 1 2 plus del tau 1 2 del x 1 into delta x 1, into delta x 2 into 1; what is this? This is tau 2 1 why it is tau 2 1 because its normal direction is 2 minus 2 actually and force acting along 1

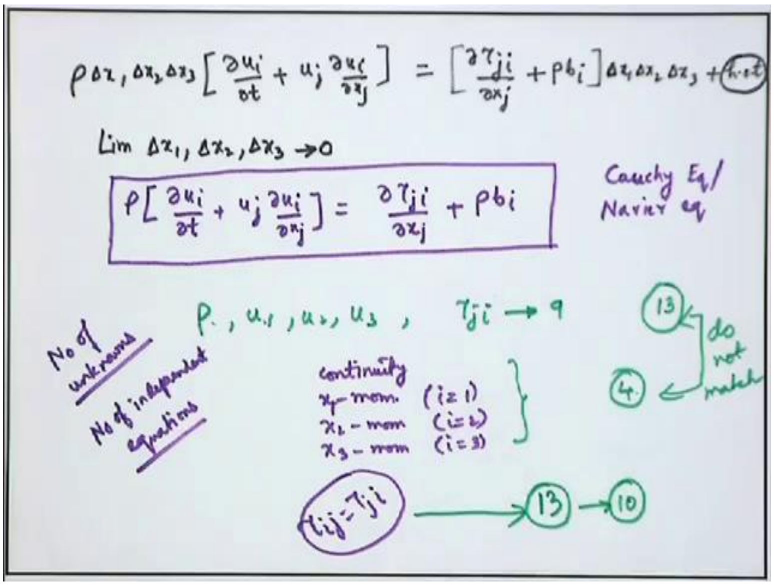
So, tau 2 1 into delta x 1 into 1, and this is tau 2 1 plus del tau 2 1 del x 2 into delta x 1 into 1. So, now, we are essentially interested about the moment of forces and as I told for calculating the moment of the forces this incremental changes will not essentially matter this changes will not essentially matter. So, essentially for calculating the moments these 2 forces are like almost equal and opposite they are actually not equal there is an incremental change, but in the limit they are almost equal opposite and anti parallel. So, they will form a couple.

So, the couple moment of these 2 forces what is that clockwise or anticlockwise you see this force is upward and this force is downward, so this is creating an anticlockwise moment, so this is a positive moment. So, this is tau 1 2 into delta x 1 into delta x 2. So, tau 1 2 into delta x 2 was the force times delta x 1 is the arm of the force, arm of the moment. So, tau 1 2 into delta x 1 into delta x 2 plus there will be higher order effect because of this we are neglecting that, then let us consider these 2 forces these 2 forces will create a clockwise moment.

So, this is minus tau 2 1 into delta x 1 into delta x 2; if you are interested you write the higher order terms also, but I am writing symbolically as higher order terms; this if there is any body force if it passes through the point O that will also not create any moment. So, even if there is a body force it is still that will not give rise to a moment that is equal to the moment of inertia. Moment of inertia is for this element is 1 by 12 into m rho delta x 1 into delta x 2, into delta x 1 square plus delta x 2 square by 12 oh sorry 12 I have already written that into theta double dot.

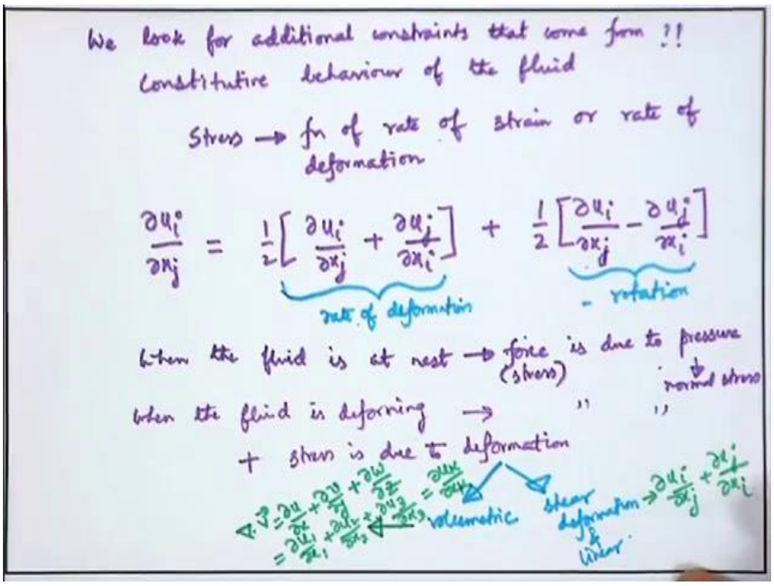
Now, take the limit as delta x 1, delta x 2 tends to 0 that is this entire element shrinks to a point. If you take this limit then what happens first of all these terms will be canceled then if you take the limit then this higher order term will tend to 0 and this terms will tend to 0 therefore, you are left with tau 1 2 is equal to tau 2 1. So, from angular momentum consideration, whatever we get we can generalize this and write tau i j is equal to tau j i. This is a very powerful relationship, but we have to keep in mind that we have made one very important assumption while deriving this; what is the assumption? The assumption that we have made is that we have neglected anybody couple; see if there is a body force it is still because its moment with respect to o is 0 until and unless it is asymmetrically distributed, but if there is a body couple then because of the body couple there could be an additional couple moment. So, the assumption is let me write it and assumption no body couple.

(Refer Slide Time: 14:59)



So, let us revisit the number of equations and the number of unknowns, I will bring the previous page. So, what you see here is that we had 13 number of unknowns and 4 number of independent equations. Now we have proven that is tau i j is equal to tau j i; with this proof the number of unknowns they get reduced, from 13 the unknowns become what. See out of this 9 tau i j now 6 are only independent, because tau i j is equal to tau j i. So, tau 1 2 is equal to tau 2 1, tau 1 3 is equal to tau 3 1, and tau 2 3 is equal to tau 3 2. So, we can say that out of this 9 unknowns we have only 6 as independent unknowns. So, from 13 we have reduced to 10, but we still have a gap between 10 equations sorry 10 unknowns and 4 independent equations.

(Refer Slide Time: 16:38)



So, we have to look for additional constraints. So, where from the additional constraints will come. So, let us discuss that. So, we look for additional constraints that come from where? From the constraints will come that will come from the constitutive behavior of the fluid. So, far we have discussed about fluids where you know we do not care whether it is Newtonian fluid or a non Newtonian fluid or whatever. Now we will specially consider some types of fluids. So, now, we have to keep in mind that for fluids that the stress is a function of rate of strain or rate of deformation; function of rate of strain or rate of deformation. So, now, rate of deformation if some of you have attended my kinematics lecture the rate of deformation this rate of deformation is is expressed by this generalized parameter del u i del x j. Now this we write as half of del u i del x j plus del u j del i, i plus half of del u i del x j minus del u j del x i. So, this is what and this is what. So, this expression is nothing very special it is a is equal to half of a plus b, plus half of a minus b, but here b is taken as a transpose. So, this is a symmetric tensor and this is an anti symmetric tensor.

This represents the rate of deformation and this represents rotation; basically minus of the rotation if you write it in the proper anticlockwise as positive whatever this physically implicates rotation. So, now, out of this velocity gradient tensor the rate of deformation. So, this is not the rate of deformation tensor this is velocity gradient tensor. So, the velocity gradient will have one part which is the rate of deformation tensor and another part which is the rotation tensor. So, out of this 2 which will contribute to the stress, is it deformation or rotation? So, I am asking this question in this chat; out of deformation and rotation which will give rise to stress out of deformation and rotation which will give rise to stress please answer this question in the chat. So, I have got already got five answers, and all this answers are correct that it is deformation that gives rise to stress.

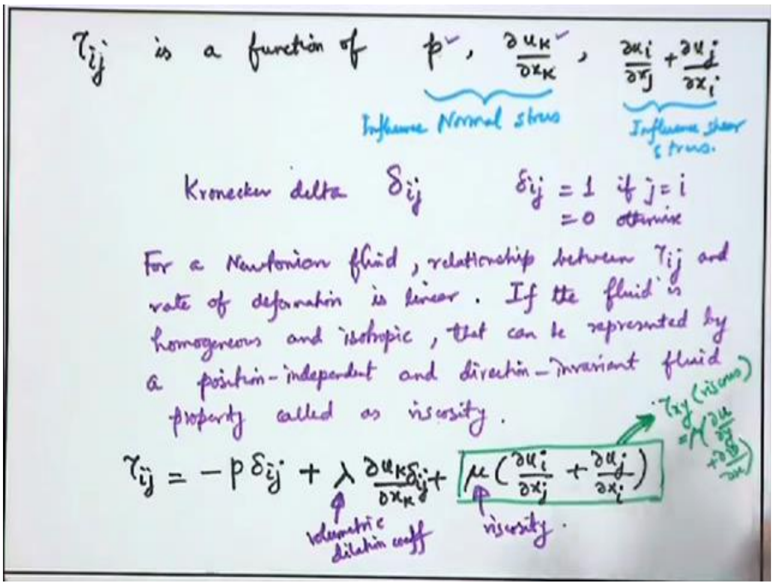
But it is also true that when the fluid is not under motion or under deformation, then also there is a stress. What physically contributes to the stress when the fluid is at rest this is my next question. So, we have now understood that when the fluid is under deformation there is a stress because of the deformation, but even if the fluid is not under deformation or under motion, there could be a source of stress what is that source of stress that is my question please give answer in the chat. So, my question is that if the fluid is under even if the fluid is under rest, it is subjected to a stress what is the source of that stress that is not deformation, but what is the source of that stress.

I have got one answer which is viscosity this is not correct, because viscosity will give rise to stress when the fluid is under deformation or motion. So, the other answer is weight that is also not correct. So, it is of course, it is it may be related to weight no doubt about it, but it is a fundamental fluid property which is not weight. Body force is another answer which is also not correct. So, I have not yet got a correct answer in the chat another answer is mass which is also not correct, gravitation force this is also not correct. So, I am looking for the correct answer and anyway let me discuss about the correct answer. So, the correct answer. So, I have not yet got correct answers, so many answers I have got in the chat, but none of these are correct. So, one answer is very close to the correct answer which is the most recent answer, any non tangential forces. So, non tangential force, but you have to be specific what is the physical origin of a non tangential force, when the fluid is at rest? It is the pressure of the fluid. So, when the fluid is at rest the force is due to pressure the stress is due to or stress is due to pressure this is the normal stress.

When the fluid is deforming, still then the pressure is present; stress is due to pressure plus stress is due to deformation. The deformation can be linear or angular linear will give rise to volumetric deformation and shear deformation, angular deformation will give rise to shear deformation. So, what is the expression for the volumetric deformation? The expression for the volumetric deformation is divergence of the velocity vector, that is del u del x plus del v del y this is the rate of volumetric strain this is the quantification of the rate of volumetric strain. So, this is del u 1 del x 1, plus del u 2 del x 2, plus del u 3 del x 3 in terms of the index notation. So, as per the index notation this is as good as del u k del x k. Remember this k index is arbitrary you can put as l m n whatever, because it is a repeated index it is a dummy index because there will be summation over this it will become number instead of the index.

Instead of the symbolic index it will become the number. So, and the shear deformation is related to del u i del x j, plus del u j del x i and. In fact, this shear deformation and also linear deformation why? Because if i and j are the same then it is del u i del x i. So, that becomes that represents linear deformation. So, this is a representative of both shear and linear deformation, because there is no guarantee that i and j are the same are different if i and j are different it will represent shear deformation, if i and j are the same it will represent linear deformation. So, let us write summarily that tau i j is a function of what.

(Refer Slide Time: 28:46)



So, tau i j is a function of pressure del u k del x k, and del u i del x j, plus del u j del x i. Out of these 2 will be giving rise to normal stress, influence normal stress and these influence shear stress . So, now, we can write therefore, tau i j as a function of this is like tau i j is related to this. Now you can write, so to isolate the normal and shear component we use a notation which is tau i j sorry which is the kronecker delta notation delta i j. delta i j is again a second order tensor is equal to 1 if j is equal to i, and is equal to 0 otherwise. So, if we use delta i j you make sure that it is only the j equal to i that is the normal component, and that is non 0 and the shear component will be 0. So, these if they are multiplied by delta i j they will give you a correct picture of the normal stress, you do not have to specify separately normal stress you just multiply this with delta i j.

Because delta i j is equal to one only if j equal to i. So, that will only have normal component; therefore, tau i j will be a function of these multiplied by delta i j plus a function of these. Now question is that what kind of function. The function can be any arbitrary function, but the dependence on the deformation if it is linear then it is called as a Newtonian fluid. So, for a Newtonian fluid relationship between tau i j and rate of deformation is linear.

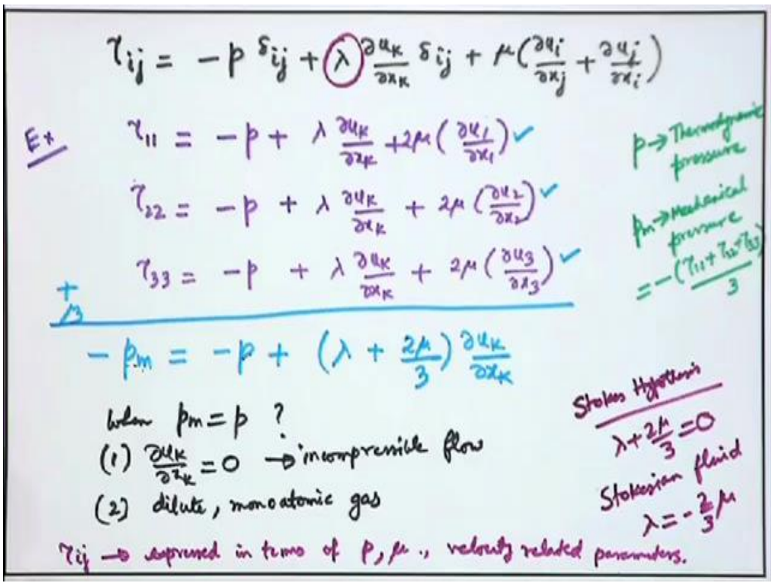
If the liquid is homogeneous and isotropic that can be represented by a homogeneous means it is independent of position, position independent and isotropic means direction invariant fluid property called as viscosity. So, let us write the expression for tau i j, this is minus P delta i j why delta i j with a normal component I have already described; why minus the minus the reason is see positive sense of pressure is compressive in nature and positive sense of normal stress is tensile in nature. So, to adjust for that there is a negative sign.

Now, it is also linearly related to the volumetric deformation. So, we put a dilation coefficient lambda, this is volumetric dilation coefficient plus mu. So, this lambda is volumetric dilation coefficient, and this mu is viscosity this is the generalized Newton's law of viscosity. Now most of you again I presume that most of you are teachers in various colleges and you teach your students nektons law of viscosity through a very simple relationship tau equal to mu d u d u i. Now suddenly you see this expression and you may have a doubt that what is then that tau equal to mu d u d i, I mean how it relates to this mu d u e d u i. So, you look into this expression and we will soon clarify that tau equal to mu d u d u i is a special case of these. So, when we say tau equal to mu d u d i first of all there is if I get back to the expression that I have written, I have missed 1 delta i j here. So, please write this delta i j.

Then tau equal to mu du du i when we say we are considering only this portion this portion. So, if you write this part, this is this will imply that tau x y just use the x y notation i equal to x and j is equal to y is equal to mu del u del y, plus del v del x. If you are considering a unidirectional flow, then there is v and there is u and no v. So, that makes tau this tau x y viscous is mu del u del y, but you have to keep in mind that there could also be a stress we will relate this volumetric dilation with the viscous effect soon, but that is an implicit thing that will come into the picture later if there is a volumetric deformation then there is an additional stress that may not be explicitly viscous stress, but there will there will be an additional stress.

So, tau equal to mu d u du i is an exclusive viscous stress representation when it is unidirectional flow, where the velocity component is along x which varies along y. So, tau there is nothing wrong with tau equal to mu d u d u i, but that is a very special case, but this is something which is a more general case. So, let us go ahead with this representation of tau i j, I am taking a fresh page and writing it again.

(Refer Slide Time: 38:18)



So, tau i j now very interesting is the normal stress. So, take an example tau 1 1 what is delta 1 1? Delta 1 1 is 1 because delta i j is 1 when i is equal to j. So, minus P plus lambda del u k del x k plus mu, plus 2 mu del u 1 del x 1; then tau 2 2 this is minus P plus lambda del u k del x k, plus 2 mu del u 2 del x 2; tau 3 3 minus P plus lambda del u k del x k, plus 2 mu del u 3 del x 3.

Now, this P what we write here is called as thermodynamic pressure, why do we call it thermodynamic pressure? We call it thermodynamic pressure because of the following reason, we call it thermodynamic pressure because this pressure will be related to the thermodynamic equation of state for example, for an ideal gas P is equal to rho r t. So that means, the equation of state relates basically pressure density and temperature and the thermodynamic pressure is that pressure which appears in the equation of state, but there is also another terminology which is called as mechanical pressure, which is defined as minus of tau 1 1, plus tau 2 2, plus tau 3 3 by 3. So, this is arithmetic average of the normal components of the stress. So, let us find the mechanical pressure. So, what we will do is we will add this and divide by 3. So, that will become minus P m is equal to minus P plus lambda 3 lambda by 3 will become lambda, plus this will be 2 mu into del u 1 del x 1 plus del u 2 del x 2 plus del u 3 del x 3. So, that is 2 mu into del u k del x k. So, that becomes 2 mu then divide by 3 becomes 2 mu by 3, del u k del x k.

So, this is an expression that relates mechanical pressure with thermodynamic pressure. So, we get a very important fundamental concept that mechanical pressure may not be equal to thermodynamic pressure right, but there are common cases when mechanical pressure is identically equal to thermodynamic pressure. So, let us write that. So, first question is why mechanical pressure and thermodynamic pressure are not the same.

So, mechanical pressure will usually reflect the translational degrees of freedom of molecules whereas, the thermodynamic pressure will reflect all sorts of degrees of freedom translational vibrational, rotational also these to represent different philosophical aspects or different conceptual aspects of the quantity pressure. Now when P m and P are identical equal, case one which is a very straightforward case that you have del u k del x k equal to 0. What is this? This is incompressible flow. So, for incompressible flow P m and P are identically equal, the second possible scenario is dilute mono atomic gas. So, for dilute mono atomic gas you have only translational degree of freedom and no other degrees of freedom and therefore, they are identically equal this term will be 0.

Otherwise it is not possible to say that they are equal, but stokes made an hypothesis which is called as stokes hypothesis, that lambda plus two-third mu is 0. If this is 0 then mechanical pressure and thermodynamic pressure are the same, and we have seen that this need not always be the case, but we will figure out that when is such a case true. First of all the fluids which obey this scenario or obey this relationship these are called as Strokesian fluids. Lambda is equal to minus two-third of mu; now let us discuss little bit about stokes hypothesis.

Now, we have to keep in mind that every fluid when subjected to a change will take some time to adapt to the change, and this time is known as relaxation time. So, how will it adapt to the change? It will adapt to the change by arriving at new thermodynamic properties. Now this time that the fluid takes to adapt to the change it is known as the relaxation time, but there can be a very interesting situation when the time over which the change on the fluid is imposed is shorter than the relaxation time, then the fluid cannot adapt to the change and then a new change by the time has come.

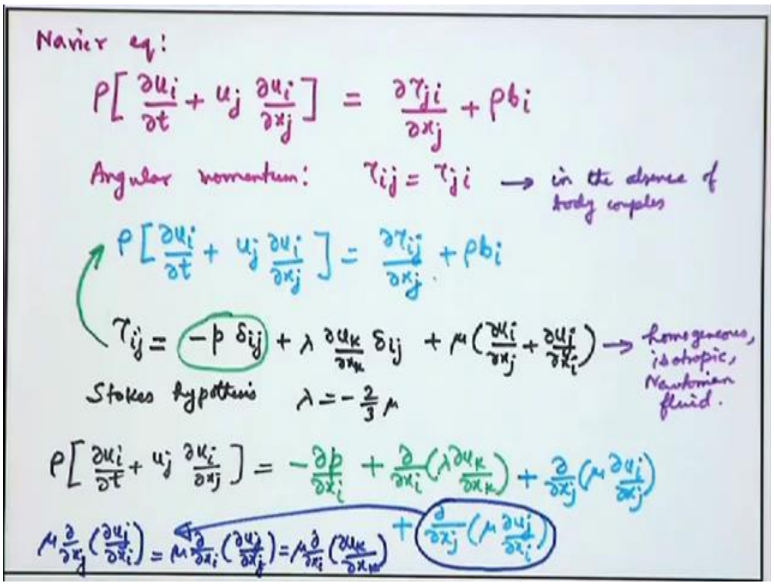
So, then we can say that it is a very rapid process; let us take an example let us consider that there is a bubble which is rapidly expanding contracting expanding contracting like this. So, if you have a bubble which is rapidly expanding and contracting then what is happening is that that the time scale of imposition of the change is very rapid, it is very fast and if it is relaxation time is not that fast, then the fluid cannot relax to a new thermodynamic stage before a new change has taken place. And then it cannot equilibrate its properties so that, it can convert all degrees of freedom to the translational degrees of freedom instantaneously.

So, if it cannot convert all degrees of freedom to the translational degrees of freedom instantaneously, then the equality of mechanical pressure and thermodynamic pressure will not take place. Therefore, for very rapid changes where the time scale of imposition of the change is faster than the relaxation time of the fluid, then the stokes hypothesis will no more be valid.

So, it is very important to understand that stokes hypothesis is not a law, it is a hypothesis. So, it does not have a proof, but there are physical arguments that for most of the practical scenarios the time scale of imposition of the disturbance is significantly larger as compared to relaxation time scale. So, that fluid relaxes almost instantaneously, so that it attains an equilibration of mechanical and thermodynamic properties, and then we can say that of course, the stokes hypothesis is a very general hypothesis, there are exceptional scenarios when the change is so fast that the stokes hypothesis is not valid, otherwise the stokes hypothesis will be valid.

So, now when the stokes hypothesis is valid, so let us come to this equation now. So, lambda is equal to. So, let us come to this equation which we have just written. So, now, in place of this lambda you write minus two-third mu, then the tau i j we are able to write in terms of how many parameters; expressed in terms of P mu and velocity related parameters. So, we have reduced the number of unknown parameter see tau i j, there were 6 unknowns and we have reduced the number of unknowns. So, now, what we will do is we will complete the derivation. So, we had a equation on tau i j for Newtonian homogeneous isotropic Newtonian and Stokesian fluid, we will substitute that in the Navier's equation.

(Refer Slide Time: 50:32)

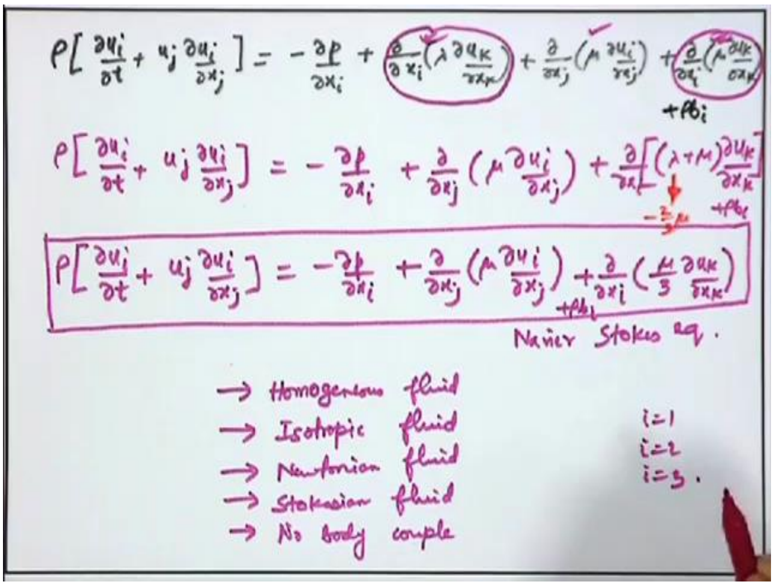


So, let us write the Navier's equation. So, this was the Navier's equation, now from angular momentum consideration tau i j is equal to tau j i. So, then this will become then we will use the expression for tau i j here, minus P delta i j plus lambda del u k del x k delta i j, plus mu del u i del x j, plus del u j del x i; and stokes hypothesis lambda is equal to minus two- third mu.

So, angular momentum this in the absence of body tuples, so I am writing the assumptions and this is for homogeneous isotropic Newtonian fluid. So, we will substitute this equation here. So, if we substitute this equation here we will get rho del u i del t, plus u j del u i del x j is equal to del tau i j del x j. So, first this term del del x j of minus p delta i j. So, delta i j will become j when j is equal to i so; that means, it will be minus del p del x i you follow it, it is minus del x j of p into delta i j, but delta i j is equal to 1 only if j is equal to i. So, that is how this j has become i; very similarly the next term is del del x i of lambda del u k del x k.

Then the next couple of terms plus del del x j of mu del u i del x j, but mu if it is homogeneous it can be taken in and out of the derivative as per your wish, plus del del x j of mu del u j del x i . So, now, we will manipulate this last term see mu is a constant. So, we can we have taken it to be function of independent of position. So, this we can write mu del del x j of del u j del x i. So, this is as good as mu del del x i of del u j del x j. Why we can write this? We can write this with an assumption of the continuity of the second order derivative. So, we it does not matter whether we differentiate with respect to x j first or with respect to x i first; and this is as good as mu del del x i of del u k del x k see this j is repeated index therefore, it is a dummy index does not matter whether you instead of j we write k l m n whatever. So, mu del del x i of del u k del x k.

(Refer Slide Time: 56:48)



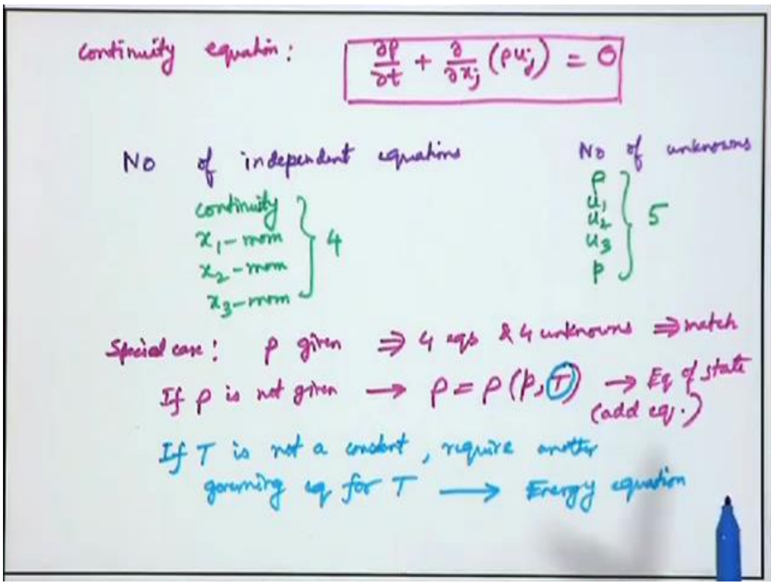
So, now we can write rho del u i del t, plus u j del u i del x j is equal to minus del P del x i plus del del x i of lambda del u k del x k, plus del del x j of mu del u i del x j plus the last term because mu is a constant we can take the mu also inside as per our wish constant means independent of position. So, the last term we can write del del x i of mu del u k del x k, again we can bring the mu into the derivative.

So, now we can combine these two terms and write the equation; rho del u i del t plus u j del u i del x j is equal to minus del p del x i, we are writing this term first plus del del x j of mu del u i del x j, plus del del x i of this term plus this term lambda plus mu del u k del x k. So that means, you have rho del u i del t, plus u j del u i del x j is equal to minus del p del x i plus del del x j of mu del u i del x j lambda plus mu what is lambda plus mu lambda is minus two- third of mu. So, del del x i of mu by 3 del u k del x k, this equation is known as Navier stokes equation.

So, what are the assumptions that we have taken for this equation? Homogeneous fluid, Isotropic fluid, Newtonian fluid, Stokesian fluid and nobody couple. So, these are the assumptions that we have made, where does the body couple come into the picture see we often Navier stokes equation we say momentum equation or momentum conservation equation, we do not say linear or angular because the fact that we have used tau i j is equal to tau j i, oh there is a body force in this equation which we have not written rho b i yes plus rho b i. So, add the body force in all the equation. So, we have assumed that tau i j is equal to tau j i. So, that is true if there is no body couple.

Now, we have derived the Navier stokes equation and this has essentially 3 components; i equal to 1, i equal to 2, i equal to 3 plus you have the continuity equation.

(Refer Slide Time: 61:20)



So, I am writing the continuity equation in the index notation, this is the continuity equation in the index notation. So, now, let us check number of equations, independent equations and number of unknowns; first let us write the unknowns. So, now, if you look at this equation I am bringing. So, first in the continuity equation what are the unknowns you see?

Let us list the unknowns rho u 1, u 2, u 3; in the Navier stokes equation you have again rho u 1, u 2, u 3 there is a additional unknown which is pressure. So, P how many unknowns are there 1, 2, 3, 4, 5, 5 unknowns what are the independent equations the continuity? X 1 momentum, x 2 momentum, and x 3 momentum. So, 4 equations, still there are number of unknowns number of equations do not close each other, in a special case they will close each other what is the special case when rho is known.

So, let us write this special case; rho is given, it may be a constant it may be a function of other variables, but if it is given then you have four equations and four unknowns that matches. So, you do not require any more equation. If rho is not given then what will rho depend on? If rho is not given you may assume that rho is a function of pressure and temperature, this is equation of state additional equation. So, you may believe that with this additional equation you close the equations with unknowns, but a very interesting thing with this additional equation you have got an additional parameter which is temperature.

So, for an isothermal case it is good enough because there the temperature is constant it is no more a variable, but if T is not a constant you require another governing equation for temperature, and then the system will close and that equation is called as energy equation. We will discuss about this energy equation in one of our upcoming courses on convective heat transfer on convection where we will discuss about this energy equation. In this brief purview of the dynamics of fluids flow we do not have the scope of discussing the energy equation, but I am just showing you that how these fluid flow equations are coming in the context of the energy equation or in other words how the energy equation is coming in the context of the fluid flow equations.

Now, a final note, so we have seen that Navier stokes equation, now when what are the characteristics or what are the specialties of the Navier stokes equation. See the Navier stokes equation is a second order partial differential equation, and it is a non-linear partial differential equation why non-linear? It is non-linear because of this term u j del u i del x j and it is a as coupled system of partial differential equations. So, that is one of the big challenges. So, you can have a linear system of equations if the left hand side of the Navier stokes equation becomes 0, and we will see in the next lecture that there are certain special cases like fully developed flow and all this we will discuss in the next lecture, when the left hand side of the Navier stokes equation becomes 0, and then that equation is called as stokes equation.

So, remember that I mean stokes was a brilliant mathematician. So, he contributed several things in fluid mechanics. So, there is something which is called as a stokes law. Stokes law is the law that describes the drag force of a slowly moving sphere in a viscous liquid. So, that is stokes law, then stokes equation stokes equation is the Navier stokes equation with this left hand side equal to 0, and stokes hypothesis that we have discussed today that is lambda is equal to minus two- third mu.

So, we need not confuse all this stokes law, stokes equation and stokes hypothesis and. So, until and unless the left hand side is 0 it is a system of coupled non-linear partial differential equations. Not only that you have pressure as a variable although you do not have an explicit governing equation for pressure. So, all this makes the solution of the Navier stokes equation very challenging until and unless you encounter very simple special cases, otherwise analytical solution may be difficult and we have to go for computational techniques. The computational techniques that solve the Navier stokes equation coupled with the continuity equation, these are known as computational fluid dynamics or c f d technique.