Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 51
Navier-Strokes equation- Part-1

Today we will discuss about dynamics of viscous flows. That is we will extend those equations for cases in which viscous affects are going to be important. So, it is first important to know that where will affect of viscous forces come in the governing equations; to understand that we will first have a general overview.

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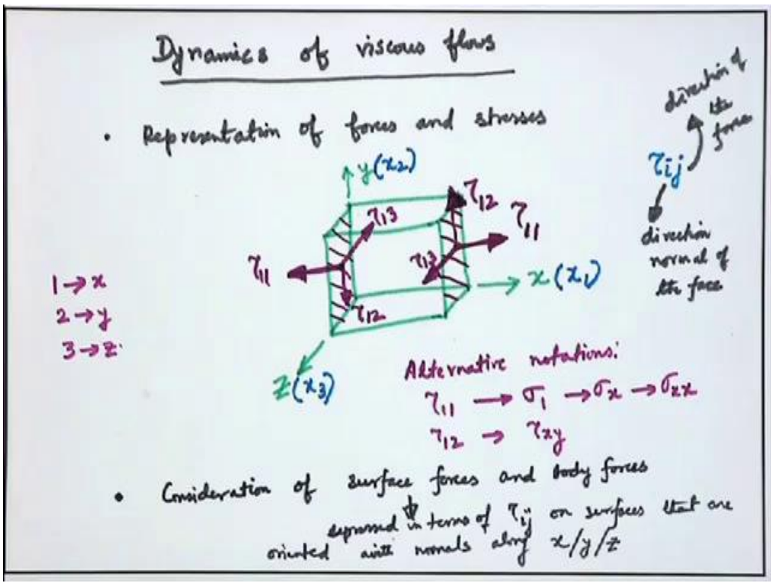

So, first we consider our broad agenda that is dynamics. So, the first agenda broad agenda first sub agenda that we will consider under the broad agent is the representation of the forces and stresses.

So, representation of forces or stresses to do that, let us consider a rectangular volume of fluid let us say this is x direction, this is y direction this is z direction in an alternative notation. This x direction is also called as x 1 direction, y direction as x 2 direction and z direction as x 3 direction. So, there are six phases on this volume element, and the specialty of each of these phases is that each of these phases has normal either along x or along y or along z. Now the forces per unit area on these are represented by the stress tensor components tau i j, what is i and what is j. So, I represent the direction normal of the phase and j represents the direction of the force.

So, let us take an example. So, this diagram is just for notation this diagram is not for calculating the actual stresses and forces and all these, this is just to explain you the notation. So, this phase will have force components, one normal force component like this. So, that force per unit area how will you represent it. So, there will be first tau then the first index is the direction normal of the phase. What is the direction normal of the phase? The direction normal of the phase is x 1. So, tau 1 the second index j is the direction of action of the force that is also along x 1, so this is tau 1 1. So, this is tau 1 1 this is the direction this is the force per unit area.

But importantly, now I will explain you why the 2 indices are required. So, at the same point now if you change the orientation of the area, if you change the orientation of the area then this normal force to the area will change right. So, if you change the orientation of the area the normal force to the area will change. So, the normal force will depend on the orientation of the area right. So, similarly the tangential force will also depend on the orientation of the area. So, tau i j the value of tau i j will depend on the orientation of the plane, that is in this case the ith direction; that is why 2 indices are required for describing the stress on the state of stress on this surface.

So, this tau i j this quantities these are not vectors, why these are not vectors? These are not vectors because vectors will require just one index for its specification, but here they require 2 indices more than one index, and the second index is there to describe the direction normal. So, these 2 indices use of these 2 indices makes this a second order tensor. Anything with 2 indices is not a anything merely with 2 indices is not qualified as a second order tensor, it also satisfies some other conditions that it maps a vector onto a vector, but we will not get into all those technicalities, because this is very introductory level course I will just discuss about the notation; and how to use this notation to derive the governing equations for viscous flows.

Now the tangential stress this is the normal stress the tangential stress. So, the tangential stress what will be this, tau what is the first index first index is the direction normal. So, direction normal is 1 and the direction of action of the force is 2, because this force is acting along y which is x 2. Similarly this is tau 1 3; now just let us consider the other phase and write the notation, for all the 6 phases it is important to write the notation, but you know the figure will be very clumsy we will write the notation here. So, here what is the direction of the normal see direction normal of a surface is designated by the outer normal. So, what is the outer normal to this surface? The outer normal to this surface is given by what? The outer normal to this surface is given by the axis x 2.

Sorry, axis x 1 minus x 1. So, the outer normal to this is axis minus x 1 that is why the normal force whenever you write this tau 1 1 actually this diagram we are using just for clarifying the notation, here if it is tau 1 it will be tau 1 1 here it will be tau 1 1 plus some change, because you have moved along x. So, this diagram is not for the change this diagram is just for describing the notation that you have to keep in mind. So, this is tau 1 1; positive tau 1 1 is along the negative 1 direction simply because the outward normal is along the negative 1 direction it will be more clear if we draw the shear stresses.

This is the positive sign convention of tau 1 2 that we draw on this, why see we are drawing tau 1 2 positive tau 1 2 along negative 2, along negative x 2 this is positive x 2 this is negative x 2, because the normal is along negative x 1. Similarly tau 1 3 the positive sign of tau 1 3 we are taking along negative x 3, the whole reason is that the outward normal is along positive along negative x 1. So, it is possible to draw the various stress tensor components tau i j and I just want to discuss alternative notations here, because different books use different notations and you may be a little bit confused with the notations.

So, alternative notations tau 1 1 its alternative notations is in somewhere is sigma 1 or sigma x or sigma x x this is normal stress, tau 1 2 tau x y like that. So, 1 for x 2 for y so, I am writing 1 for x in the index notation, 2 for y, and 3 for z. So, tau 1 2 is like tau x y tau 1 1 is like sigma 1 why one only one index may be sufficient because it is just the same index again, and that is sigma x or sigma x x. So, different books you will we will use different notations and it is important that you get familiar with the equations.

So, the first point is representation of forces and stresses. So, we have represented the stresses, and the next is consideration of surface forces and body forces. So, surfaces forces we have expressed in terms of the stress tensor components, and body forces. So, surface forces expressed in terms of tau i j on surfaces that are oriented with normal’s along either x or y or z. In addition to the surface forces you also talk about body forces, and we have discussed earlier that how to account for the body forces.

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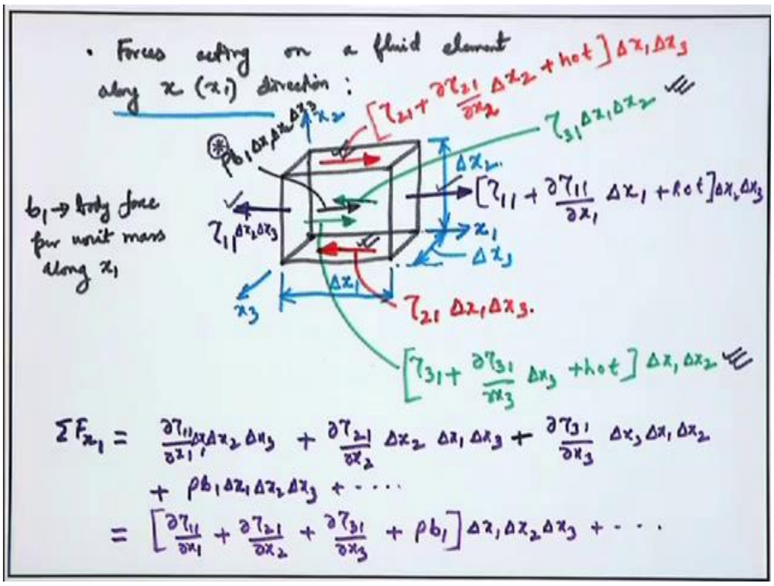

Next we will consider the forces acting on a fluid element first is x or x 1 direction, forces acting on a fluid element along x or x 1 direction. So, let us draw a fluid element this is x 1, this is x 2 this is x 3. So, in this element let us say that this is delta x 1, this is delta x 2 and this is delta x 3. So, our job is to represent the forces along the x 1 direction. So, there are six phases first we will consider the surfaces forces, there are six phases we will consider the surfaces forces along the x 1 direction. So, we will start with the left phase. So, let this be the origin we will start first with the left phase. So, on this on the left phase remember that tau i j, i does not represent the direction it is the j that represents the direction. So, for all this cases you want j equal to 1, because you want force along x 1 direction. So, no matter whatever is i you are looking for the forces which have the second index or the stresses which have the second index as j equal to 1.

Now, on the left phase this normal phase is represented by what? Tau the direction normal is 1 and the force the direction of the force is also one. So, this is just the representation of the stress the force is stress times the area, the area of this phase is delta x 2 into delta x 3. On the opposite phase see now we have to really quantify here, on the opposite phase here if it is tau 1 1 here what is the value? Tau 1 1 plus delta tau 1 1 del x 1 into delta x 1. Again I am repeating the previous diagram was just a diagram to clarify the notation, but this diagram is to calculate the actual forces. So, we have to calculate the changes. So, this is the new tau 1 1 of course plus higher order terms times delta x 2 into delta x 3.

So, on the left phase this is on the right phase, and then let us consider the bottom phase and the top phase. On the bottom phase the force along x 1 direction what will be its corresponding stress? Tau i j out of that j is 1 sorry yes j is 1 because you are considering the force along x 1 direction; what is i for this phase, i for this phase is 2 actually in the negative 2 direction, but in the index you do not write minus. So, it is tau tau 2 1, tau 2 1 the first index is for the normal direction second index is for the direction of force and this times delta x 1 into delta x 3.

Then on the top phase see the direction of tau 2 1 positive direction positive sense is opposite, because here the direction normal is positive x 2 here the direction normal is negative x 2; that is why this is along positive x 1 this is along negative x 1. So, what is this? Tau 2 1 plus del tau 2 1 del x 1 into delta x 2 sorry del tau 2 1 del tau x 2 see the change here is along x 2. So, that is why del tau 2 1 del x 2, plus higher order term into what is the area delta x 1 into delta x 3 then front and back.

So, first let us write the back. So, I will use a different color let me find out a color we will use the green color. So, back phase. So, we have written left right, bottom top then back and front. The back phase has outward normal along negative x 3. So, the force on the back phase this, what will be this tau the direction normal is 3, direction of force is 1 and it is negative direction by convention because the outward normal is along negative x 3 into the area of the back phase is delta x 1 into delta x 2. Then we come to the front phase, the front phase it is tau 3 1 if the back phase is tau 3 1 front phase is del tau 3 1 plus del tau 3 1 del x 3 into delta x 3, plus higher order term into delta x 1 into delta x 2 these are the surface forces.

In addition to the surface forces there are body forces. So, let us say that b 1 is a body force per unit mass along x 1. So, because the mass is rho delta x 1 into delta x 2 into delta x 3, the force is rho b 1 delta x 2, delta x 1, delta x 2, delta x 3. So, this diagram it is not a complete free body diagram, but this diagram shows all forces on a fluid element along x 1 direction. So, now, let us write an expression for the resultant force along the x 1 direction. So, the resultant force along the x 1 direction, first let us consider the difference between the left phase and the right phase that is this and this. So, if you consider the difference tau 1 1 and tau 1 1 they are oppositely oriented they will cancel.

So, del tau 1 1 del x 1 into delta x 2 into delta x 3, higher order terms we will write symbolically at the end then that into delta x sorry into delta x 1 is also there. So, delta x 1 so then after that there will be a resultant force. So, this is on this 2 phases then we will consider the top and the bottom. So, this one and this one; so the resultant of this 2 will be del tau 2 1 del x 2, into delta x 2 into delta x 1 into delta x 3 and finally, the back and the front. So, the back and the front will be this one and this one.

So, plus sorry plus del tau 3 1 del x 3, into delta x 3 into delta x 1 into delta x 2, these are the surface forces then there is a body force, this there is a body force. So, the body force is plus rho b 1 plus delta x 1 delta x 2 delta x 3, plus higher order terms. So, this we can write as del tau 1 1 del x 1, plus del tau 2 1 del x 2, plus del tau 3 1 del x 3, plus rho b 1 into delta x 1 into delta x 2 into delta x 3 plus higher order terms, this is the resultant force along x 1 direction.

So, we will use a new page and we will write the expression again, because we will generalize it.

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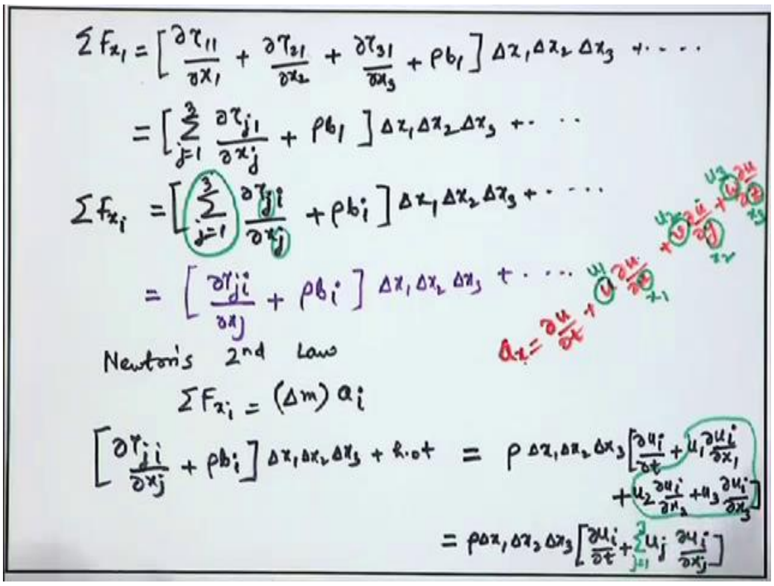

So, resultant force along x 1 I am writing the previous expression again del tau 1 1 del x 1, plus del tau 2 1 del x 2, plus del tau 3 1 del x 3, plus rho b 1 into delta x 1, delta x 2 delta x 3 plus higher order terms. So, this you can write as summation of del tau j 1 del x j plus rho b 1, this j is from 1 to 3. So, this is force along x 1. So, what will be the force along x i direction; where i can be anything 1, 2 or 3 just replace 1 with i. So, then this becomes summation of j is equal to 1 to 3 del tau j i del x j, plus rho b i delta x 1, delta x 2, delta x 3 plus higher order terms.

Now, we have to remember one thing that in this expression there is a repeated index j and there is summation over the repeated index. Now Einstein introduced a notation that if there is a repeated index there must be an invisible summation over the repeated index; that means, the summation one should not the sigma one may write, but it is not necessary that one will write sigma. So, instead of this it is as good as del tau j i del x j plus rho b i. Now let us apply Newton's second law to this motion to the fluid element. So, Newton's second law resultant force along x i is equal to the mass times acceleration along i, acceleration along i means basically acceleration along x i.

So, resultant force let us write del tau j i del x j, plus rho b i into delta x 1, delta x 2, delta x 3 plus higher order terms; delta m is the mass of the fluid element that is rho into delta x 1 into delta x 2 into delta x 3, times acceleration along i. Let us write the acceleration expression del u i del t plus u 1 del u 1 del u i, del x 1, plus u 2 del u i del x 2, plus u 3 del u i del x 3. See you may be surprised that how I have written this expression; just let us try to see how we have used this notation, let me write the expression for acceleration along x this you are familiar with.

Acceleration along x is del u del t, plus u del u del x plus v del u del y, plus w del u del z. So, acceleration now the notation is instead of u we are calling u 1, instead of x we are calling x 1. So, instead of v you are calling u 2, instead of y we are calling x 2, instead of w we are calling u 3, instead of z we are calling x 3. So, this is acceleration along x instead of that it is x i, u becomes u i. So, that expression is here. So, this expression with u equal to u i is here.

So, how can you write this in a compact notation? So, we have to see how can we write this in a compact notation. So, del u i del t, plus u j del u i del x j. So, how why it is? See actually there is an invisible summation over this for j is equal to 1 to j then that is the expression which is here, but as I told if there is a repeated index, you need not write this sigma. So, it is as good as u j del u i del x j, because this j is repeated in the expression it means implicitly that there is summation of j from 1 to j.

So, now we can write we I am rewriting the previous expression in the new page with the left hand side written in the right and right hand side written in the left this is the standard fluid mechanics convention the acceleration we write in the left and the forces in the right.

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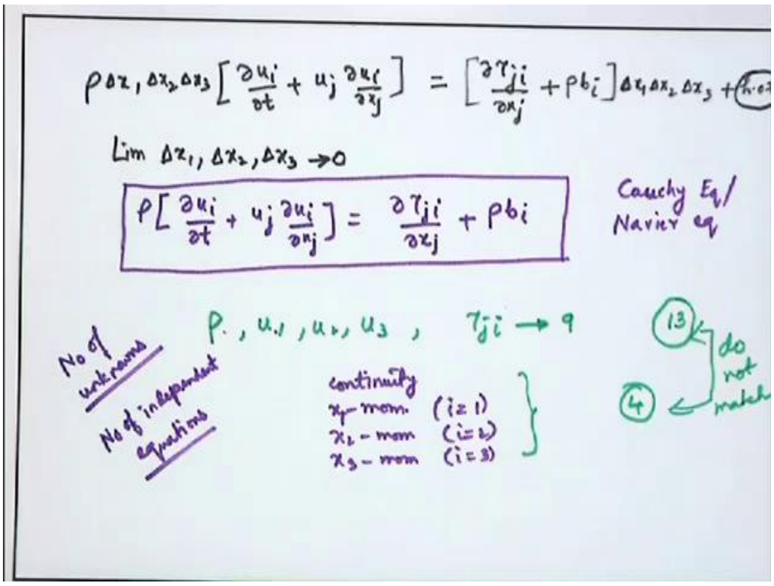

So, rho delta x 1, delta x 2, delta x 3 del u i del t plus u j del u i del x j is equal to del tau j i del x j, plus rho b i into delta x 1 into delta x 2 into delta x 3, plus higher order terms. Now what is our objective? Our objective is to basically write a differential equation at a point. So, when we are interested to write the differential equation at a point, we will take the limit as delta x 1, delta x 2, delta x 3 tending to 0, so that you know the entire volume of fluid shrinks to a point. So, if you take that limit then all the higher order terms will vanish, and then we are left with the following equation; this is a very general equation, this equation is known as the Cauchy equation or the Navier equation.

So, let me write it here, now this equation is very general because we have not assumed any type of fluid. So, this is absolutely the most general equation or motion that you can think of. But if you do not consider most special versions and just consider this generic version, you may not be able to close a problem and solve it; why? Let us look into the number of independent equations available with you and number of unknowns. So, number of unknowns let us first write; what are the unknowns let us identify the unknowns. Rho is one unknown in general density is unknown, in a special case density may be known, but in general it is also a variable then u 1, u 2, u 3 then tau j i; how many such components are there j is equal to 1 2 3 and i equal to 1 2 3, so 9 components.

So, 9 plus 1 plus 2 plus 3 plus 4 9 plus 4 total 13; 13 unknowns number of independent equations; one equation is the continuity equation which we discussed in the course of kinematics of flow, and then the x momentum or x 1 momentum that is i equal to 1, x 2 momentum that is i equal to 2, and x 3 momentum that is i equal to 3. So, total you have 4 independent equations. So, four independent equations and 13 unknowns- these do not match.

So, we require additional equations or additional constraints to close the system. And our subsequent discussion will be in a quest to get these additional equations so that we can reach this gap between the numbers of equations, the independent equations and the number of unknown.