Introduction to Fluid Mechanics

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Lecture – 49
Application of RTT: Conservation of angular momentum

We will go ahead with the description of the Reynolds transport theorem in an arbitrarily moving reference frame.

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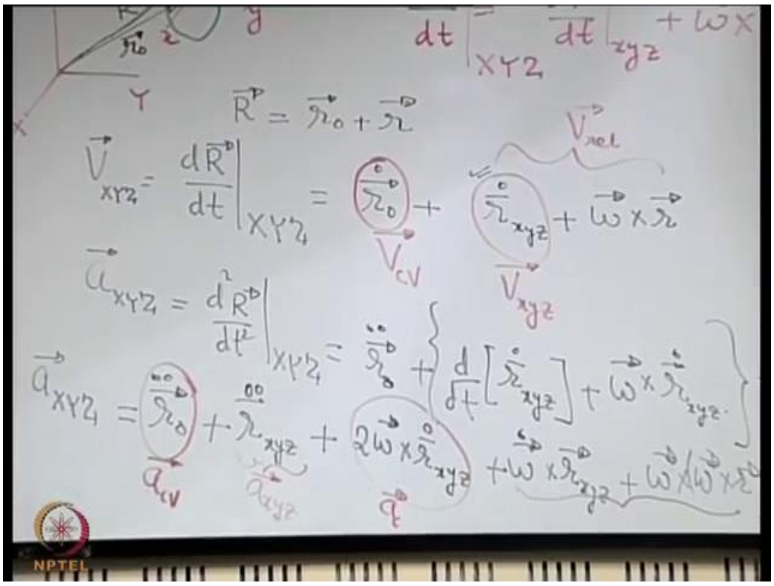

So, in the last class we were discussing about the derivative of a vector in a arbitrarily moving reference frame. And the result that we saw that if you have 2 reference frames one is capital X Y Z another reference frame is small x y z, where small x y z may be having an arbitrary motion in terms of an angular velocity in terms of a linear velocity which may even vary with time and so on. So, if you have the vector A which is there in the small x y z reference frame, the derivative of that with respect to the capital X Y Z which is like a stationary reference frame is this plus omega cross A

Clearly, if this small x y z reference frame is having no omega, then the derivatives are identical. Now what we will do is we will try to utilize this to figure out that what happens with respect to the velocities and accelerations of fluid elements located in a controlled volume, because of the moment of the control volume. So, now, we are going to consider that the control volume is not stationary, it is moving. And when we say that the control volume is moving let us say that we consider the control volume to be such that small x y z is attached to the control volume. So, the way in which the control volume moves small x y z motion represents that. So, the small x y z motion has 2 important aspects. One is the translatory motion another is the rotational motion. So, the translatory motion is like if you have the origin of the small x y z as this one, then if you have the position vector say let us let us call it r naught, the rate of change of this r naught with respect to time gives the translatory velocity of the small x y z reference frame. On the top of that small x y z reference frame is having a rotational velocity in general. There may be special cases when rotational velocity is not there.

Now, if you consider say a point in a control volume let us say a point P this point in general represents point where the fluid has a velocity acceleration and so on. So, it is a point in the flow field. So, this point in terms of it is position vector is described by this position vector which we say call as capital R, but if we are trying to analyze everything with respect to a small x y z reference frame, for us the important quantity or the important vector is the position vector of the point P relative to the origin of the small x y z reference frame and let us say that is small r. So, capital R is the position vector of the same point P relative to capital X Y Z origin, and small r with respect to small x y z origin.

So, directly from vector addition we can write capital R is equal to r naught plus small r. Now what we will do in the next step is we will try to find out the velocity. So, the velocity at the point P in an absolute sense is d R d t that is d capital R d t with respect to capital X Y Z, this is the velocity with respect to capital X Y Z. So, this is just in a short hand notation we may not always write d d t, but they are write a dot on the top to indicate that it is a time derivative. So, this is r naught dot plus it is d d t of small r relative to capital X Y Z d d t of small r relative to capital X Y Z is, d d t of small r relative to small x y z plus omega cross small r from this theorem so; that means, this is as good as r dot small x y z plus omega cross r.

So, if you try to figure out that what are the implications of these different terms, let us just try to understand this is nothing, but the velocity of the control volume. The control volume is having a translational velocity; this is the translational velocity of the control volume. Because this is the time rate of change of position vector of the origin of the control volume of the of the origin of the reference frame describing the control volume.

And this is what this is the velocity as visualized from the reference frame small x y z, and this is of course, omega cross r. So, this 2 terms together may be sort of thought of as a relative velocity. So, this is the absolute velocity this is the velocity of the control volume, this is the velocity relative to the control volume like that. So, the relative velocity has one component because of the translation another component because of the rotation. And if we want to find out the acceleration, acceleration is what is the soul important quantity for us, because by that we can relate with the newton's second law of motion.

So, acceleration with respect to capital X Y Z; that means, you have to differentiate these again with respect to time. So, let us identify the different terms and write the first term it will become r naught double dot right. Next r dot small x y z, just substitute that with a in this expression. So, the d d t of this one is what we are looking for. So, what is d d t of this one, that is d d t of r dot small x y z plus omega cross r dot small x y z. That is for the second term. For the third term again the same formula we can apply capital A replace by omega cross r. So, that will be d d t of omega cross r relative to small x y z plus omega cross omega cross r. So, next stage is just the simplification and collection of the relevant terms.

So, the first term r naught double dot, plus r small x y z double dot, plus you have omega cross r dot, but let us simplify another term before writing that. Let us simplify this one, this you can differentiate just as if it is like a product rule, but maintaining the order because for the cross product the order is important. So, it is omega dot cross r small x y z, plus omega cross r dot small x y z. Just like the product rule. So, if you collect now, omega cross r dot small x y z and this omega dot omega cross r dot small x y z. So, it will become 2 omega cross r dot small x y z. Then you have omega dot cross r small x y z plus omega cross omega cross r.

So, if we just give the interpretation as we did for the velocity for the different terms, what is r naught double dot. It is the acceleration of the control volume linear acceleration. So, this is the acceleration of the control volume. Then what is r double dot relative to small x y z. This is the acceleration of the particle under consideration relative to the control volume. And then if you consider these 2 terms, these are angular acceleration sort of that is if you have angular acceleration effect. So, this is this is not directly angular acceleration effect, but because of the rate time rate of change of the even a fixed vector because of the angular motion within the element. So, if you have a even a fixed vector because of the rotation that vector is getting changed with respect to time. And this is directly because of the angular acceleration of the reference frame. So, these are directly as a consequence of the angular motion of the control volume. So, the angular velocity and angular acceleration. These are direct consequences of the linear, this is the direct consequence of the linear acceleration of the control volume. This is because of the linear acceleration of the particle relative to the control volume. And this is a combination of the linear and angular motion effect.

So, as you all know this is called the Coriolis component of acceleration. So, why where from this Coriolis component of acceleration comes. So, if you have a reference frame, that rotates and relative to that reference frame something translates. So, the combination of that gives rise to an acceleration. So, the 2 things which are necessary for this. One is the reference frame should be a rotating reference frame, another thing is that there should be a translatory motion relative to the reference frame. And then it will experience a sort of acceleration, and that acceleration we will try to deflect the particle from it is original locus.

And it is very common in particle mechanics also in fluid mechanics just think about ocean currents. So, earth is like a rotating object and over the earth the ocean currents are moving. So, you have the water moving in a sea with a particular velocity on the top of a reference frame which is rotating. And that is why and the rotational sense is different at different places. So, you will see that there will be a certain deflection of the ocean current in the northern hemisphere. And in the southern hemisphere these things you have studied in junior classes of geography, but these are these are very important examples of the implications of Coriolis effects in fluid mechanics.

Now, what we can do is we can just write it in a bit of a more compact way, we can write that, So, this term we will write as a small x y z. And all the remaining terms like this, this we know this is like a Coriolis and this is the angular velocity f a angular velocity and angular acceleration f a.

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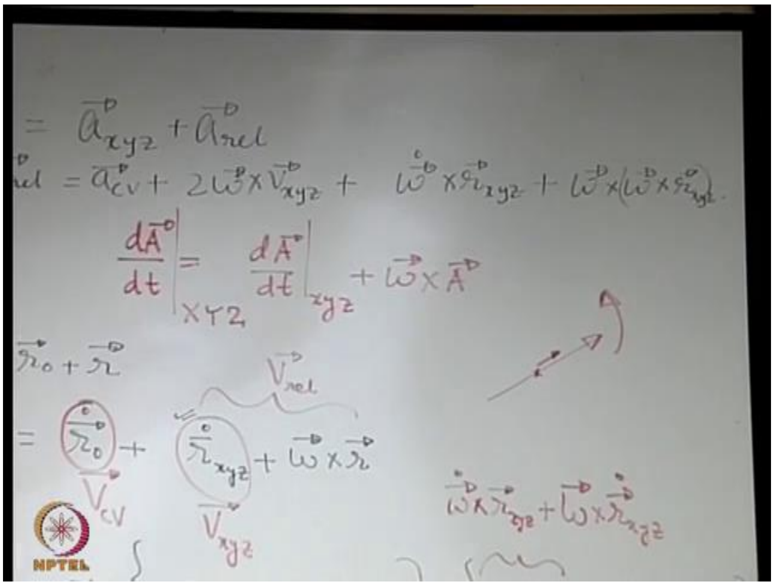

So, we will just write a capital X Y Z as if is equal to a small x y z plus. Let us give the all other terms a name a relative this is just a name that we are giving. So, in the a relative we have a control volume plus 2 omega cross v x y small x y z plus omega dot cross r small x y z, plus omega cross omega cross r, r is again small x y z. So, you can clearly see that, this type of transformation from reference frame which is stationary to arbitrarily moving reference frame. And when you do that the transformation terms appear in the form of the quantities, which you are visualizing relative to the moving reference frame. Because when you are writing your equations of motion relative to the moving reference frame all quantities you are measuring relative to the moving reference frame. So, that position vector, the velocity vector all those things you are measuring. So, you can clearly see that when the transformation is made in the right hand side everything is written relative to small x y z. So, small x y z has become your reference.

Now how will this be important in the context of use of the Reynolds transport theorem. Let us try to go back to that. Let us say that we are interested to write Reynolds transport theorem for linear momentum conservation. So, Reynolds transport theorem applied for linear momentum conservation. Let us say that we are interested to write the theorem relative to this control volume, this green colored contour is the surface control surface of the control volume. And that is our basis for which we are writing the theorem.

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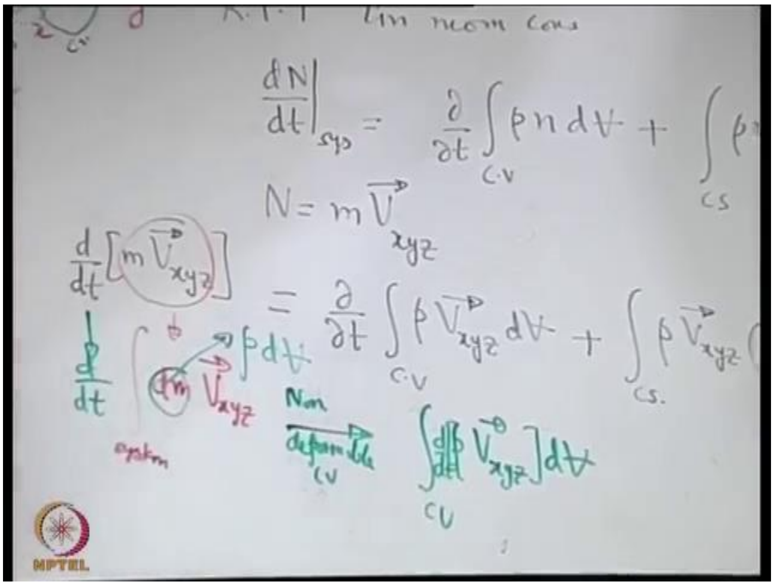

Let us first write the general theorem d N d t system we recall that capital N is the quantity that we are interested to conserve, it may be a scalar it may be a vector, small n is capital N per unit mass. And this eta is the unit vector normal to the area that elemental area that is considered for writing the area of plus term.

Now, one important thing is that we have no restriction on the choice of this capital N and small n. That is we are not restricted to write this only as quantity is relative to an inertial reference frame. We may even write this relative to quantities in a non-inertial reference frame. Because when we derive this theorem, we never had any consideration for a specific inertial or non-inertial choice of the reference frame. So, we can as well write say capital N as say m into V small x y z, as an example nothing restricts us from doing that because it is just some quantity that we are looking for it may be linear momentum relative to any reference frame. So, when we write that, left hand side becomes d d t of m. Let us complete first write the right hand side and then we will see that whether the way in which we have written the left hand side is the proper way or not. We have to keep in mind that this V relative is nothing, but V small x y z that is the velocity of the fluid relative to the control volume. So, this is V small x y z dot.

Now, let us devote our concentration on the left hand side. When you look into the left hand side, see when we write this what is the assumption? The assumption is that the entire system is having the velocity v small x y z relative to the small x y z, but that is how mass into this one. Because this is the total linear momentum of the system. So, total linear momentum of the system if we write that as mass of the system times this velocity; that means, we are implicitly assuming that the entire mass of the system is having this velocity, which is not correct. Because fluid is a in is a is a deformable medium and in general you can have different velocities at different points.

So, it is not appropriate to write it in this way, but write it as a integral form of like integral of d m V small x y z over the system. The reason is quite clear that at different points in the system it might have a different velocity. So, when you want to represent the total linear momentum physically we are representing the total linear momentum. The linear momentum consideration should keep in mind that there could be different velocities at different points in the system. And then we may write this as d m as how do we replace d m in terms of the elemental volume. So, this is the rho into an elemental volume d V.

Next step. So, if we put d d t outside then you have the d d t also. Next step can we put the d d t inside yes or no, we have discussed about this thing earlier. See what is the variable with respect to which integration is done volume. So, if that volume is not a function of time then we may put it inside the integral without requiring to put any correction terms. So, we are assuring that it is a non-deformable control volume. So, with a non-deformable control volume we may write it as integral of say rho V small x y z d d t of this one, d V. See this assumption is not a very restricted assumption, because for most of the problems that we are considering this it is not. So, common to have control volume which is arbitrarily accelerating plus deforming, but I can give you a nice example if you are fascinated with mechanics try with this example. Say you have a balloon in a balloon you fill it up with water. Just throw it with a spin and make the water come out of that. And try to figure out how the velocity of the water coming out is changing with time. So, you require everything. A deformable control volume a arbitrarily accelerating because the balloon when the water is coming out the balloon might be having arbitrary rotation and arbitrary linear motion. And in an elementary level I mean this problem experimentally one can do, but theoretically I would say it is one of the very tough problems in mechanics. Which will require the combination of the understanding of the fluid mechanics and solid mechanics and deformable control volume, arbitrarily accelerating control volume, it will test your understanding of mechanics to the best of abilities.

Let us not go into that complication here. So, we assume that this is what we are writing. Next step is that we will assume that rho is not changing with time. So, if rho is not changing with time, we will just write this as integral of rho d d t of v small x y z, d V. The other thing is let see magically we were as if we transformed it from the system to the control volume. That we will not do immediately, but we should keep in mind that when we are writing in a limiting sense, that is we are dividing it by delta t and in the limit delta t tends to 0. We can write it in the limiting sense tending to a control volume, because in the limit as delta t tends to 0, system tends to control volume that is what we have experienced earlier, but that we will do in a bit of later stage. Till now we will preserve it as a system sense why, the next step will be giving you a clue of why we are trying to do that.

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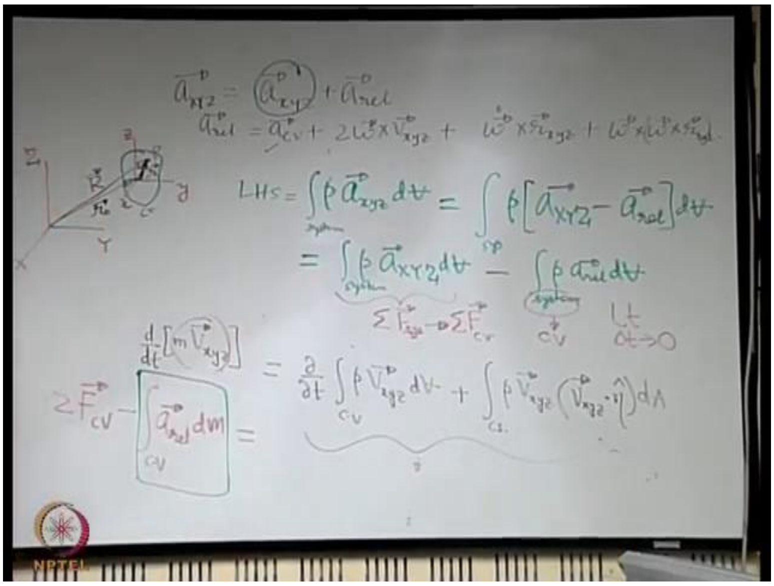

So, in the next step what we will do? We will write the left hand side simplification. This is again system we have to keep in mind that what are the assumption what we made. So, you have the rho and what is d d t of V small x y z this is the,

Student: acceleration.

Acceleration small x y z what is appearing here. So, this is rho acceleration small x y z d V system. What is our trouble is that when we are having acceleration relative to small x y z, we cannot directly apply the newton's second law of motion for that? Because newton's second law of motion is valid for an inertial reference frame. Small x y z in a special case when it is moving with a uniform linear velocity still an inertial reference frame, but in a general case not. So, to address the generality we have to keep in mind that, we need to first transform it to capital X Y Z, so that for that we can use the newton's second law of motion.

So, we will write this as integral of rho in place of a small x y z, we will write a capital X Y Z minus a relative. So, when you take this integral of rho a capital X Y Z d V for the system and minus then, what are the implications of this 2 terms? The implications are very straightforward. The first term represents the mass into acceleration of the system as an affect the integrated the total effect mass into acceleration of the system as viewed from a inertial reference frame. So, by newton's second law of motion this, you can write as resultant force acting on the system. And since we considered the limit as delta t tends to 0, this is as good as resultant force on the control volume. And again in that same limit you have in that limit this also the system tends to control volume keeping in mind the limit that we are considering for this time derivatives that automatically implies that the delta t is tending to 0.

So, what is the final form of the equation that we are getting? Let us write it in the final form. So, the final form is, the left hand side becomes what the left hand side becomes resultant force acting on the control volume minus let us say integral of rho d V you can say as d m elemental mass. So, it is just like here also. So, it is like integral of a relative d m is equal to the right hand side whatever is there. So, if you just look into it from a user viewpoint or a formula user view point, what is the correction that has occurred because of the transformation? The correction has occurred mainly with one important thing, that you have this correction term. And this correction term is because of the acceleration of the control volume. Because if you see all the terms are related to acceleration. This is directly related to the acceleration of the control volume. This wherever there is angular velocity you expect that it is related to that. So, angular velocity angular acceleration and again angular velocity. So, these are all related to an the nature of an accelerating reference frame.

So, if the reference frame is not of accelerating nature, then or if the if the reference frame is not having an angular velocity also, because having an angular velocity automatically makes it of accelerating nature. At least the last term is important. So, when none of these terms are important, then this will be 0 we will see when and in the right hand side you see that nothing great has happened, we have replaced in all the terms the absolute velocity by the relative velocity.

So, when you are writing the Reynolds transport theorem for linear momentum conservation, then basically all the absolute velocities are replaced with the relative velocities. This must anyway always relative velocity, but the other terms these are now replaced by relative velocity. And in the left hand side there is a correction because of the non-inertial effect of the reference frame just like minus mass into acceleration. This is just like a pseudo force. So, with that understanding let us try to solve a few problems to understand that what we do for a moving reference frame in reality.